ON THE FIRST GROUP OF THE CHROMATIC COHOMOLOGY OF GRAPHS

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ABSTRACT.

The algebra of truncated polynomials $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ plays an important role in the theory of Khovanov and Khovanov-Rozansky homology of links. We have demonstrated that Hochschild homology is closely related to Khovanov homology via comultiplication free graph cohomology. It is not difficult to compute Hochschild homology of $\mathcal{A}_m$ and the only torsion, equal to $\mathbb{Z}_m$, appears in gradings $(i, \frac{m(i+1)}{2})$ for any positive odd i. We analyze here the grading of graph cohomology which is producing torsion for a polygon. We find completely the cohomology $H^{1,v-1}_{\mathcal{A}_2}(G)$ and $H^{1,2v-3}_{\mathcal{A}_3}(G)$. The group $H^{1,v-1}_{\mathcal{A}_2}(G)$ is closely related to the standard graph cohomology, except that the boundary of an edge is the sum of endpoints instead of the difference. The result about $H^{1,v-1}_{\mathcal{A}_2}(G)$ gives as a corollary a fact about Khovanov homology of alternating and $+$ or $-$ adequate link diagrams. The group $H^{1,2v-3}_{\mathcal{A}_3}(G)$ can be computed from the homology of a cell complex, $X_{\Delta,4}(G)$, built from the graph G. In particular, we prove that $\mathcal{A}_3$ cohomology can have any torsion. We give a simple and complete characterization of those graphs which have torsion in cohomology $H^{1,2v-3}_{\mathcal{A}_3}(G)$ (e.g. loopless graphs which have a 3-cycle). We also construct graphs which have the same (di)chromatic polynomial but different $H^{1,2v-3}_{\mathcal{A}_3}(G)$. Finally, we give examples of calculations of width of $H^{1,*}_{\mathcal{A}_3}(G)$ and of cohomology $H^{1,(m-1)(v-2)+1}_{\mathcal{A}_m}(G)$ for $m > 3$.

Contents

1. Introduction

In 1945, G. Hochschild[1] defined a homology of any ring [Hoch]. The algebra of truncated polynomials $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ was one of the first to have its Hochschild homology computed. This algebra plays an important role in the theory of Khovanov and Khovanov-Rozansky homology of links. The only torsion in the Hochschild homology $HH_{i,j}(\mathcal{A}_m)$ is equal to $\mathbb{Z}_m$ and it appears

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[1]According to [Mac], p.237: "...From this result [that if a space has all homotopy groups, but the first, G, trivial then homology of the space does not depend on the choice of the space representative so is a property of the group G alone] developed the surprising idea that cohomology, originally studied just for spaces, could also apply to algebraic objects such as groups and rings. Given his topological background and enthusiasm, Eilenberg was perhaps the first person to see this clearly. He was in active touch with Gerhard Hochschild, who was then a student of Chevally at Princeton. Eilenberg suggested that there ought to be a cohomology (and a homology) for algebras. This turned out to be the case, and the complex used to describe the cohomology of groups (i.e., the bar resolution) was adapted to define the Hochschild cohomology of algebras. Eilenberg soon saw other possibilities for homology, and he and Henri Cartan wrote the book Homological algebra, which attracted lively interest among algebraists such as Kaplansky. A leading feature was the general notion of a resolution, say of a module M; such a resolution was an exact sequent of free modules F_i ... $\rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M$. Earlier work by Hilbert on syzygies suggested this idea; an essential feature was a theorem comparing two resolutions used to prove that the cohomology they give is (up to isomorphism) independent of the choice of the resolution".

in gradings $(i, \frac{m(i+1)}{2})$ for any odd i (see Section 2.1). We have demonstrated in [Pr-2] that Hochschild homology is closely related to Khovanov homology via comultiplication free graph cohomology described in [H-R-1, H-R-2]. In this paper we analyze graph cohomology in gradings which have torsion for a polygon (as we know from the relation with Hochschild homology). More precisely we are interested in the group $H_{\mathcal{A}_m}^{1,(m-1)(v-1)-\frac{(m-2)(n-1)}{2}}(G)$ for some odd n; $v = v(G)$ denotes the number of vertices of G (Section 2). In the case of $n = 3$ the corresponding cochain complex has length two, $C^{0,(m-1)(v-2)+1} \xrightarrow{\partial^0} C^{1,(m-1)(v-2)+1} \to 0$, and we find cohomology completely for $m = 2$ and $m = 3$, relating graph cohomology to a cohomology of a cell complex built from the graph G (Theorem 4.1). For the case of $m = 2$, which corresponds to classical Khovanov homology, we obtain an interesting corollary about Khovanov homology of alternating and adequate links (related to results of [Shu, A-P]). For the case of $m = 3$ we show that the related cohomology $H_{\mathcal{A}_3}^{1,2v-3}(G)$ can have arbitrary torsion and we characterize those graphs which have torsion in $H_{\mathcal{A}_3}^{1,2v-3}(G)$. We illustrate our results by several corollaries and by a few examples. In the sixth section we construct graphs which have the same chromatic (even dichromatic and Tutte) polynomial but different $H_{\mathcal{A}_3}^{1,2v-3}(G)$. In relation to these examples we also analyze A_3 chromatic graph cohomology for one vertex and two vertices product. We prove, in particular, that $H_{\mathcal{A}_3}^{1,2v-3}(G)$ is a 2-isomorphism (matroid) graph invariant. Finally, in the eight section, we report several computations concerning width of $tor(H_{\mathcal{A}_3}^{1,*}(G))$. We venture also into $H_{\mathcal{A}_m}^{1,(m-1)(v-2)+1}(G))$ calculations for $m > 3$ and state several questions and conjectures.

2. Basic facts about Hochschild homology and chromatic graph cohomology

We recall in this section definitions of Hochschild homology and chromatic graph cohomology and the relation among them [Pr-2]. We follow [Lo, H-R-2, H-P-R, Pr-2] in our exposition.

2.1. **Hochschild homology.** Let k be a commutative ring and $\mathcal{A}$ a k-algebra (not necessarily commutative). Let $\mathbb{M}$ be a bimodule over $\mathcal{A}$, that is, a k-module on which $\mathcal{A}$ operates linearly on the left and on the right in such a way that $(am)a' = a(ma')$ for $a, a' \in \mathcal{A}$ and $m \in \mathbb{M}$. The actions of $\mathcal{A}$ and k are always compatible (e.g. $m(\lambda a) = (m\lambda)a = \lambda(ma)$). When $\mathcal{A}$ has a unit element 1 we always assume that $1m = m1 = m$ for all $m \in \mathbb{M}$. Under this unital hypothesis, the bimodule $\mathbb{M}$ is equivalent to a right $\mathcal{A} \otimes \mathcal{A}^{op}$-module via $m(a' \otimes a) = ama'$. Here $\mathcal{A}^{op}$ denotes the opposite algebra of $\mathcal{A}$, that is, $\mathcal{A}$ and $\mathcal{A}^{op}$ are the same as sets but the product $a \cdot b$ in $\mathcal{A}^{op}$ is the product ba in $\mathcal{A}$. The product map of $\mathcal{A}$ is usually denoted $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\mu(a, b) = ab$.

In this paper we work only with unital algebras (the algebras of truncated polynomials in most cases). We also assume, unless otherwise stated, that $\mathcal{A}$ is a free k-module, however in most cases, it suffices to assume that $\mathcal{A}$ is k-projective, or less restrictively, that $\mathcal{A}$ is flat over k. Throughout the paper the tensor product $\mathcal{A} \otimes \mathcal{B}$ denotes the tensor product over k, that is, $\mathcal{A} \otimes_k \mathcal{B}$.

Definition 2.1 ([Hoch, Lo]). *The Hochschild chain complex $C_*(\mathcal{A}, \mathbb{M})$ of the algebra $\mathcal{A}$ with coefficients in $\mathbb{M}$ is defined as:*

$$\ldots \xrightarrow{b} \mathbb{M} \otimes \mathcal{A}^{\otimes n} \xrightarrow{b} \mathbb{M} \otimes \mathcal{A}^{\otimes n-1} \xrightarrow{b} \ldots \xrightarrow{b} \mathbb{M} \otimes \mathcal{A} \xrightarrow{b} \mathbb{M}$$

where $C_n(\mathcal{A}, \mathbb{M}) = \mathbb{M} \otimes \mathcal{A}^{\otimes n}$ and the Hochschild boundary is the k-linear map $b : \mathbb{M} \otimes \mathcal{A}^{\otimes n} \to \mathbb{M} \otimes \mathcal{A}^{\otimes n-1}$ given by the formula $b = \sum_{i=0}^{n}(-1)^i d_i$, where the face maps d_i are given by
$d_0(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)$,
$d_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n)$ for $1 \leq i \leq n - 1$,
$d_n(m, a_1, \ldots, a_n) = (a_n m, a_1, \ldots, a_{n-1})$.

In the case when $\mathbb{M} = \mathcal{A}$ the Hochschild complex is called the cyclic bar complex.
By definition, the nth Hochschild homology group of the unital k-algebra $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathbb{M}$ is the nth homology group of the Hochschild chain complex denoted by $H_n(\mathcal{A}, \mathbb{M})$. In the particular case $\mathbb{M} = \mathcal{A}$ we write $C_*(\mathcal{A})$ instead of $C_*(\mathcal{A}, \mathcal{A})$ and $HH_*(\mathcal{A})$ instead of $H_*(\mathcal{A}, \mathcal{A})$.

The algebra $\mathcal{A}$ acts on $C_n(\mathcal{A}, \mathbb{M})$ by $a \cdot (m, a_1, ..., a_n) = (am, a_1, ..., a_n)$. If $\mathcal{A}$ is a commutative algebra then the action commutes with boundary map b. Therefore, $H_n(\mathcal{A}, \mathbb{M})$ (in particular, $HH_*(\mathcal{A})$) is an $\mathcal{A}$-module.

If $\mathcal{A}$ is a graded algebra, $\mathbb{M}$ a coherently graded $\mathcal{A}$-bimodule, and the boundary maps are grading preserving, then the Hochschild chain complex is a bigraded chain complex with $(b : C_{i,j}(\mathcal{A}, \mathbb{M}) \to C_{i-1,j}(\mathcal{A}, \mathbb{M}))$, and $H_{**}(\mathcal{A}, \mathbb{M})$ is a bigraded k-module. In the case of abelian $\mathcal{A}$ and $\mathcal{A}$-symmetric $\mathbb{M}$ (i.e. $am = ma$), $H_{**}(\mathcal{A}, \mathbb{M})$ is bigraded $\mathcal{A}$-module. The main examples coming from the knot theory are $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ and $\mathbb{M}$ the ideal in $\mathcal{A}_m$ generated by x^{m-1}. In the case of $\mathbb{M} = \mathcal{A}_m$ we have (see for example [Lo]).

Proposition 2.2.

$$HH_{i,j}(\mathcal{A}_m) = \begin{cases} \mathbb{Z}_m & for & i \text{ odd}, \ j = \frac{i+1}{2}m \\ \mathbb{Z} & for & i = j = 0 \ \text{ or } \ i \geq 0 \ \text{ and} \\ & & \lfloor \frac{i}{2} \rfloor m + 1 \leq j \leq \lfloor \frac{i}{2} \rfloor m + m - 1 \\ 0 & otherwise \end{cases}$$

Here, $\lfloor x \rfloor$ denotes the integer part of x.
In particular, for i odd $HH_{i,*}(\mathcal{A}_m)$ is $\mathcal{A}_m$-module isomorphic to
$\mathbb{Z}[x]/(x^m, mx^{m-1})\{m\frac{i-1}{2} + 1\}$, where $\{k\}$ denotes the shift by k in the grading.

2.2. **Chromatic graph cohomology.** Chromatic graph cohomology was introduced in [H-R-1] as a comultiplication free version of Khovanov cohomology of alternating links, where alternating link diagrams are translated to plane graphs (Tait graphs). Being free of topological restrictions, chromatic graph cohomology was extended in [H-R-2] to any commutative algebra $\mathcal{A}$. We showed in [Pr-2] that $\mathcal{A}$ graph cohomology (that is, chromatic graph cohomology with underlining algebra $\mathcal{A}$) can be interpreted as a generalization of Hochschild homology from a polygon to any graph. We have this interpretation only for a commutative $\mathcal{A}$. It seems to be, that if one works with general graphs and not necessary commutative algebras then these algebras should satisfy some "multiface" properties. Very likely planar algebras or operads provide the proper framework.

Definition 2.3. *For a given commutative k-algebra $\mathcal{A}$, symmetric $\mathcal{A}$-module M and a graph G with a base vertex v_0, we define M-reduced $\mathcal{A}$ graph cochain complex and cohomology as follows (see [H-R-2, Pr-2] for details).*

(i) *The cochain k-modules $C^i(G, v_0)$ are defined as follows: $C^i(G, v_0) = \oplus_{|s|=i, s \subset E(G)} C_s^i(G, v_0)$. The k-module $C_s^i(G, v_0) = M \otimes \mathcal{A}^{k(s)-1}$, where $k(s)$ is the number of components of the graph $[G : s]$ which is the subgraph of G containing all vertices of G and edges s. We visualize the product $M \otimes \mathcal{A}^{k(s)-1}$ as attachment of M to a component of $[G : s]$ containing v_0 and attachment of $\mathcal{A}$ to any other component of $[G : s]$.*

The cochain map $d^i : C^i(G, v_0) \to C^{i+1}(G, v_0)$ is defined as follows: $d^i = \sum_{e \notin s} (-1)^{t(s,e)} d_e^i$ where d_e^i depends on whether e connects different components of $[G : s]$ or it connects vertices in the same component of $[G : s]$. In the last case we assume d_e^i to be the identity map. If e connects different components of $[G : s]$ then either

(m) *If e connects the components of $[G : s]$ containing v_0 with another components, say the first one, then*
$d_e^i(m, a_1, a_2, ...a_{k(s)-1}) = (ma_1, a_2, ...a_{k(s)-1})$,

(a) *if e connects two components not containing v_0, say the first and the second, then*
$d_e^i(m, a_1, a_2, ...a_{k(s)-1}) = (m, a_1 a_2, ...a_{k(s)-1})$.

(ii) *We define $\mathbb{M}$-reduced cohomology denoted by $H_{\mathcal{A}, \mathbb{M}}^*(G, v_0)$ as the cohomology of the above cochain complex. If we assume $\mathbb{M} = \mathcal{A}$ we obtain $\mathcal{A}$-cohomology of graphs, $H_{\mathcal{A}}^*(G)$ (often called the chromatic graph cohomology as it categorifies the chromatic polynomial of G [H-R-2]).*

Remark 2.4. *The boundary map $d_1 : \mathbb{M} \otimes \mathcal{A} \to \mathbb{M}$ in Hochschild homology is the zero map for a commutative algebra $\mathcal{A}$ and a symmetric module $\mathbb{M}$ $(d_1(m, a) = ma - am = 0)$ and thus $H_0(\mathcal{A}, \mathbb{M}) = \mathbb{M}$. Therefore, it is convenient to consider the variant of chromatic graph cohomology,*

$\hat{H}^*_{\mathcal{A},\mathbb{M}}(G, v_0)$, which has the zero map in place of Khovanov comultiplication (see [H-R-2, Pr-2]) as defined below[2]:

(i) Consider the cochain complex of a graph $(\hat{C}^*_{\mathcal{A},\mathbb{M}}(G, v_0); \hat{d})$ obtained by modifying $(C^*_{\mathcal{A},\mathbb{M}}(G, v_0); d)$ as follows: $\hat{C}^i_{\mathcal{A},\mathbb{M}}(G, v_0) = C^i_{\mathcal{A},\mathbb{M}}(G, v_0)$, $\hat{d}_e = d_e$ for a state s such that e is connecting different components of $[G : s]$ but if e has endpoints on the same component of $[G : s]$ we put $\hat{d}_e = 0$. The cohomology of the cochain complex $(\hat{C}^*_{\mathcal{A},\mathbb{M}}(G, v_0); \hat{d})$ will be denoted by $\hat{H}^*_{\mathcal{A},\mathbb{M}}(G, v_0)$. For $\mathbb{M} = \mathcal{A}$ we write simply $\hat{H}^*_{\mathcal{A}}(G)$. This version of chromatic graph cohomology was considered in [H-R-2].

(ii) Let $\ell(G)$ denote the girth of the graph G, that is, the length of the shortest cycle in G. Then straight from the definition of $H^i_{\mathcal{A},\mathbb{M}}(G, v_0)$ and $\hat{H}^i_{\mathcal{A},\mathbb{M}}(G, v_0)$ we get:

$\hat{H}^i_{\mathcal{A},\mathbb{M}}(G, v_0) = H^i_{\mathcal{A},\mathbb{M}}(G, v_0)$ for $i < \ell(G) - 1$, and torsion part satisfies (for k being a principal ideals domain)

$tor(\hat{H}^i_{\mathcal{A},\mathbb{M}}(G, v_0)) = tor(H^i_{\mathcal{A},\mathbb{M}}(G, v_0))$ for $i = \ell(G) - 1$.

2.3. $\mathcal{A}$ graph cohomology of a polygon as Hochschild homology of $\mathcal{A}$.

We use the following result connecting Hochschild homology and chromatic graph cohomology observed in [Pr-2] (because Theorem 2.5 concerns only a polygon we can work with any, not necessary commutative, algebra $\mathcal{A}$).

Theorem 2.5. Let $\mathcal{A}$ be a unital algebra which is a free k-module[3], $\mathbb{M}$ an $\mathcal{A}$-bimodule and P_{n+1} – the $(n + 1) - $gon. Then for $0 < i \leq n$ we have:

$$\hat{H}^i_{\mathcal{A},\mathbb{M}}(P_{n+1}) = H_{n-i}(\mathcal{A}, \mathbb{M}).$$

Furthermore, if $\mathcal{A}$ is a graded algebra and $\mathbb{M}$ a coherently graded module then $\hat{H}^{i,j}_{\mathcal{A},\mathbb{M}}(P_{n+1}) = H_{n-i,j}(\mathcal{A}, \mathbb{M})$, for $0 < i \leq n$ and every j.

In particular, $\hat{H}^{n,j}_{\mathcal{A},\mathbb{M}}(P_{n+1}) = (sign?)H_{0,j}(\mathcal{A}, \mathbb{M}) = \mathbb{M}/(am - ma)$

Corollary 2.6. $\hat{H}^{i,j}_{\mathcal{A}}(P_{n+1}) = HH_{n-i,j}(\mathcal{A})$, for $0 < i \leq n$ and every j. Furthermore, for a commutative $\mathcal{A}$, $H^{i,j}_{\mathcal{A}}(P_{n+1}) = \hat{H}^{i,j}_{\mathcal{A}}(P_{n+1})$, for $0 < i < n$ and $H^{n,*}_{\mathcal{A}}(P_{n+1}) = 0$, $\hat{H}^{n,*}_{\mathcal{A}}(P_{n+1}) = HH_{0,*}(\mathcal{A}) = \mathcal{A}$. For a general $\mathcal{A}$, $\hat{H}^{n,*}_{\mathcal{A}}(P_{n+1}) = HH_{0,*}(\mathcal{A}) = \mathcal{A}/(ab - ba)$.

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[2]The modification allows us a concise formulation of Theorem 2.5 even for a nonabelian $\mathcal{A}$.

[3]We assume in this paper that $\mathcal{A}$ is a free k-module, but we could relax the condition to have $\mathcal{A}$ to be projective or, more generally, flat over a commutative ring with identity k; compare [Lo]. We require $\mathcal{A}$ to be a unital algebra in order to have an isomorphism $\mathbb{M} \otimes_{\mathcal{A}^\epsilon} \mathcal{A}^{\otimes n+2} = \mathbb{M} \otimes \mathcal{A}^{\otimes n}$; the isomorphism is given by $\mathbb{M} \otimes_{\mathcal{A}^\epsilon} \mathcal{A}^{\otimes n+2} \ni (m, a_0, a_1, ..., a_n, a_{n+1}) \rightarrow (a_{n+1}ma_0, 1, a_1, ..., a_n, 1)$ which we can write succinctly as $(a_{n+1}ma_0, a_1, ..., a_n) \in \mathbb{M} \otimes \mathcal{A}^{\otimes n}$. We should stress that in $\mathbb{M} \otimes_{\mathcal{A}^\epsilon} \mathcal{A}^{\otimes n+2}$ the tensor product is taken over $\mathcal{A}^\epsilon = \mathcal{A} \otimes \mathcal{A}^{op}$ while in $\mathbb{M} \otimes \mathcal{A}^{\otimes n}$ the tensor product is taken over k.

We work in the paper with $\mathbb{M} = \mathcal{A} = \mathcal{A}_m$ but in the future analysis for $M = (x^{m-1})$, $\mathcal{A} = \mathcal{A}_m$ will be given.

2.4. **Interesting gradings for $\mathcal{A}_m$-algebras.** The relation between Hochschild homology and graph cohomology of a polygon allowed us to find graph cohomology of a polygon for algebras $\mathcal{A}_m$. In particular, the torsion of $H^{i,j}_{\mathcal{A}_m}(P_v)$ is supported by (i,j) such that $v - i$ is even and $mi + 2j = mv$. We have:

Corollary 2.7 ([Pr-2]).

$$tor(H^{i,j}_{\mathcal{A}_m}(P_v)) = \begin{cases} H^{i,j}_{\mathcal{A}_m}(P_v) = \mathbb{Z}_m & for \ v - i \ even, \ 0 < i \leq v - 2, \ j = \frac{v-i}{2}m \\ 0 & otherwise \end{cases}$$

The study of torsion in $H^{v-2,m}_{\mathcal{A}_m}(G)$ was initiated in [H-P-R]. In this paper we concentrate on the first cohomology $H^{1,j}_{\mathcal{A}_m}(G)$, partially motivated by the fact that computing whole $H^{*,*}_{\mathcal{A}_m}(G)$ is NP-hard (so, up to famous conjecture, has exponential complexity) while computing $H^{1,j}_{\mathcal{A}_m}(G)$ for a fixed m has polynomial complexity.

Corollary 2.7 (applied for $i = 1$) suggest also that if a graph, G has an odd cycle of length n then the grading $j = (v - n)(m - 1) + \frac{n-1}{2}m$ should be of considerable interest. We decided to work with the case $n = 3$, that is, to analyze $H^{1,(v-2)(m-1)+1}_{\mathcal{A}_m}(G)$.

It is well known that $H^{1,j}_{\mathcal{A}_2}(G)$ is trivial for $j > v - 1$, in fact the whole first cohomology is supported by $j = v - 1, v - 2$ Also for $m = 3$ the highest possible grade in $H^{1,*}_{\mathcal{A}_3}$ is equal to $2v - 3$, that is, $H^{1,j}_{\mathcal{A}_3}(G) = 0$ for $j > 2v - 3$. This gives another reason to concentrate on $j = 2v - 3$ grading. We can ask whether for general m, the value $j = (v-2)(m-1)+1$ is the highest possible grading of nonzero $H^{1,j}_{\mathcal{A}_m}(G)$. As a first step in this direction, sufficient for $m = 3$, we prove the following proposition generalizing slightly Corollary 13 (1c) of [H-P-R].

Proposition 2.8. *Assume that $m \geq 3$ and $0 < i < \ell(G)$, then $H^{i,j}_{\mathcal{A}_m} = 0$ for $j \geq (m - 1)(v - i)$*

Proof. We use the following notation (following that of [Vi-1]). Any $s \in E(G)$ is called a state of G and in the case of $\mathcal{A} = \mathcal{A}_m$, an enhanced state S is a state s with every component of $[G : s]$ decorated by a weight x^i, $0 \leq i < m$. Notice first that for $i < \ell(G)$, and $j > (m - 1)(v - i)$ we have $C^{i,j}_{\mathcal{A}_3}(G) = 0$ and furthermore $C^{i,(m-1)(v-i)}_{\mathcal{A}_3}(G)$ is freely generated by enhanced states S, with underlining state s, and such that each component of $[G : s]$ has weight x^{m-1}. For $i > 0$ and $m > 2$, consider a component of $[G : s]$, say X_e which has an edge e. Consider the state $s' = s - e$ and the weights of components of $[G : s']$ in which one component of $X_e - e$ has weight x and the other x^{m-2}. Because $m - 2 \geq 1$, therefore the image of this enhanced state is the chosen generator S of $C^{i,(m-1)(v-i)}_{\mathcal{A}_3}(G)$. Thus d_{i-1} is an epimorphism and $H^{1,(m-1)(v-i)}_{\mathcal{A}_3} = 0$. $\square$

We will discuss further improvements of Corollary 13 (1c) of [H-P-R] in the sequel paper (compare examples in Section 7). In particular we show that $H^{1,j}_{\mathcal{A}_5}(G) = 0$ for $j > 4v - 7$.

2.5. **From cohomology to homology.** In Sections 3 and 4 we are performing very concrete calculations and we observe that it is much easier to work (visualize the chain complex) in the homology case. Homology and cohomology of a chain complex are related by the universal coefficient theorem. We give here the simplified version in the form we use (see for example [Hat]).

Proposition 2.9. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated then*

$$H^n(C;\mathbb{Z}) = H_n(C;\mathbb{Z})/tor(H_n(C;\mathbb{Z})) \oplus tor(H_{n-1}(C;\mathbb{Z})).$$

In particular, in the cases we consider mostly, we have:
$H^{0,(m-1)(v-2)+1}_{\mathcal{A}_m}(G) = H^{\mathcal{A}_m}_{0,(m-1)(v-2)+1}(G)/tor(H^{\mathcal{A}_m}_{0,(m-1)(v-2)+1}(G))$,
$H^{1,(m-1)(v-2)+1}_{\mathcal{A}_m}(G) = H^{\mathcal{A}_m}_{1,(m-1)(v-2)+1}(G) \oplus tor(H^{\mathcal{A}_m}_{0,(m-1)(v-2)+1}(G))$
and $H^{\mathcal{A}_m}_{1,(m-1)(v-2)+1}(G) = ker(C^{\mathcal{A}_m}_{1,(m-1)(v-2)+1}(G) \to C^{\mathcal{A}_m}_{0,(m-1)(v-2)+1}(G))$ is a free abelian group.

Because our (co)chain groups are free and finitely generated, we use the same enhanced states to describe basis of chain, $C^{\mathcal{A}_m}_{i,j}(G)$, and cochain, $C^{i,j}_{\mathcal{A}_m}(G)$, groups. Matrices describing chain and cochain maps are transpose one to another in these bases. In the cochain map we use multiplication in algebra so in chain map we use dual comultiplication. The concrete cases will be described in detail in next sections.

3. THE CASE OF $\mathcal{A}_2 = \mathbb{Z}[x]/(x^2)$ AND KHOVANOV HOMOLOGY

In this section we compute $H^{1,v-1}_{\mathcal{A}_2}(G)$ for every graph, showing, in particular, that if G is connected then the torsion part of $H^{1,v-1}_{\mathcal{A}_2}(G)$ is either trivial if G is bipartite or otherwise (i.e. if G has an odd cycle) it is equal to $\mathbb{Z}_2$. In particular, the version of Shumakovitch's conjecture for graphs holds for the height one (that is, $H^{1,*}_{\mathcal{A}_2}(G)$ has no $\mathbb{Z}_4$ in its torsion part). The detailed analysis of $H^{2,v-2}_{\mathcal{A}_2}(G)$ will be given in the sequel paper.

Theorem 3.1. *Let G be a simple graph then*

(0) $H^{0,v-1}_{\mathcal{A}_2}(G) = \mathbb{Z}^{p^{bi}_0}$, *where p^{bi}_0 is the number of bipartite components of G.*

(1) $H^{1,v-1}_{\mathcal{A}_2}(G) = \mathbb{Z}^{p_1-(p_0-p^{bi}_0)} \oplus \mathbb{Z}^{p_0-p^{bi}_0}_2$, *where p_0 is the number of components of G and $p_1 = rank(H_1(G,\mathbb{Z})) = |E| - v + p_0$ is the cyclomatic number of G.*

Corollary 3.2. *If G is a connected simple graph then*

$$H_{\mathcal{A}_2}^{1,v-1}(G) = \begin{cases} \mathbb{Z}^{p_1} & \text{if } G \text{ is a bipartite graph} \\ \mathbb{Z}^{p_1-1} \oplus \mathbb{Z}_2 & \text{if } G \text{ has an odd cycle} \end{cases}$$

Proof. As mentioned in Section 2, it is easier to visualize the chain complex in the homology case. That is, consider the interesting for us part of the chromatic graph chain complex

$$0 \leftarrow C_{0,v-1}^{\mathcal{A}_2}(G) \xleftarrow{d_1} C_{1,v-1}^{\mathcal{A}_2}(G) \leftarrow 0.$$

We have $C_{0,v-1}^{\mathcal{A}_2}(G) = \mathbb{Z}^v$ and enhanced states forming the basis of $C_{0,v-1}^{\mathcal{A}_2}(G)$ can be identified with vertices of G, namely to an enhanced state in which every vertex but v_i has attached the weight x and v_i has weight 1 we associate the vertex v_i. We also have $C_{1,v-1}^{\mathcal{A}_2}(G) = \mathbb{Z}^E$ and enhanced states forming the basis of $C_{1,v-1}^{\mathcal{A}_2}(G)$ can be identified with edges of G (we write E for cardinality $|E|$ as long as the meaning is clear). The matrix describing the map d_1 is the incidence matrix of the (unoriented) graph G (see for example [Big]). That is the image of an edge is equal to the sum of its endpoints. Therefore, for a bipartite graph $H_{i,v-1}^{\mathcal{A}_2}(G) = H_i(G,\mathbb{Z})$. However, any odd cycle identifies a vertex on the cycle with its opposite. It easily leads for a connected simple graph to

$$H_{0,v-1}^{\mathcal{A}_2}(G) = \begin{cases} \mathbb{Z} & \text{if } G \text{ is a bipartite graph} \\ \mathbb{Z}_2 & \text{if } G \text{ has an odd cycle} \end{cases}$$

and for any simple graph to

$$H_{0,v-1}^{\mathcal{A}_2}(G) = \mathbb{Z}^{p_0^{bi}} \oplus \mathbb{Z}_2^{p_0-p_0^{bi}}.$$

Expressing Euler characteristic in two ways we obtain:
$rank(H_{0,v-1}^{\mathcal{A}_2}(G)) - rank(H_{1,v-1}^{\mathcal{A}_2}(G)) = rank(C_{0,v-1}^{\mathcal{A}_2}(G)) - rank(C_{1,v-1}^{\mathcal{A}_2}(G)) = v - E$.

Therefore, for a connected simple graph G:

$$H_{1,v-1}^{\mathcal{A}_2}(G) = \begin{cases} \mathbb{Z}^{E-v+1} = \mathbb{Z}^{p_1} & \text{if } G \text{ is a bipartite graph} \\ \mathbb{Z}^{E-v} = \mathbb{Z}^{p_1-1} & \text{if } G \text{ has an odd cycle} \end{cases}$$

and for any simple graph

$$H_{1,v-1}^{\mathcal{A}_2}(G) = \mathbb{Z}^{E-v+p_0^{bi}} = \mathbb{Z}^{p_1-(p_0-p_0^{bi})}.$$

Corollary 3.2 and Theorem 3.1 follow almost immediately from the above results and Proposition 2.9.

$\square$

3.1. From Kauffman states on link diagrams to graphs and surfaces.
A Kauffman state s of a link diagram D is a function from the set of crossings of D to the set $\{+1, -1\}$. Equivalently, to each crossing of D we assign a marker according to the following convention:
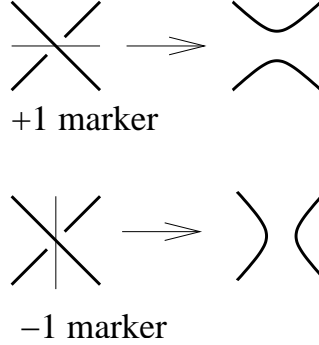
+1 marker

−1 marker

Fig. 3.1; markers and associated smoothings

By D_s we denote the system of circles in the diagram obtained by smoothing all crossings of D according to the markers of the state s, Fig. 3.1. By $|s|$ we denote the number of components of D_s. The positive state s_+ (respectively the negative state s_-) is the state with all positive markers (resp. all negative markers).

For every Kauffman state s of a link diagram D we construct a planar graph $G_s(D)$. The graphs corresponding to states s_+ and s_- are of particular interest. If D is an alternating diagram then $G_{s_+}(D)$ and $G_{s_-}(D)$ are the plane graphs first constructed by Tait from checkerboard coloring of regions of $R^2 - D$.

Definition 3.3.

(i) *Let D be a diagram of a link and s its Kauffman state. We form a graph, $G_s(D)$, associated to D and s as follows. Vertices of $G_s(D)$ correspond to circles of D_s. Edges of $G_s(D)$ are in bijection with crossings of D and an edge connects given vertices if the corresponding crossing connects circles of D_s corresponding to the vertices[4].*

(ii) *In the language of associated graphs we can state the definition of adequate diagrams as follows: the diagram D is +-adequate (resp. −-adequate) if the graph $G_{s_+}(D)$ (resp. $G_{s_-}(D)$) has no loops.*

In this language we can recall the result about torsion in Khovanov homology [A-P]; Theorem 2.2, which we generalize.

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[4]If S is an enhanced Kauffman state of D then, in a similar manner, we associate to D and S the graph $G_S(D)$ with signed vertices. Furthermore, we can additionally equip $G_S(D)$ with a cyclic ordering of edges at every vertex following the ordering of crossings at any circle of D_s. The sign of each edge is the label of the corresponding crossing. In short, we can assume that $G_S(D)$ is a ribbon (or framed) graph, and that with every state we associate a surface $F_s(G)$ whose core is the graph $G_s(D)$. $F_s(G)$ is naturally embedded in R^3 with $\partial F_s(G) = D$. For $s = \vec{s}$, that is, D is oriented and markers of $\vec{s}$ agree with orientation of D, $G_s(D)$ is the Seifert graph of D and $F_s(G)$ is the Seifert surface of D obtained by Seifert construction. We do not use this additional data in this paper but it may be of great use in analysis of Khovanov homology (compare [Pr-1]).

Theorem 3.4. [A-P]
Consider a link diagram D of N crossings. Then

(+) *If D is $+$-adequate and $G_{s_+}(D)$ has a cycle of odd length, then the Khovanov homology has $\mathbb{Z}_2$ torsion. More precisely, $H_{N-2,N+2|s_+|-4}(D)$ has $\mathbb{Z}_2$ torsion,*

(-) *If D is $-$-adequate and $G_{s_-}(D)$ has a cycle of odd length, then $H_{-N,-N-2|s_-|+4}(D)$ has $\mathbb{Z}_2$ torsion.*

In [H-P-R] we proved the following relation between graph cohomology and classical Khovanov homology of alternating links.

Theorem 3.5. *Let D be the diagram of an unoriented framed alternating link and let G be its Tait graph (i.g. $G = G_{s_+}(D)$). Let ℓ denote the girth of G, that is, the length of the shortest cycle in G. For all $i < \ell - 1$, we have*

$$H_{\mathcal{A}_2}^{i,j}(G) \cong H_{a,b}(D)$$

with $\begin{cases} a = E(G) - 2i, \\ b = E(G) - 2V(G) + 4j, \end{cases}$

where $H_{a,b}(D)$ are the Khovanov homology groups of the unoriented framed link defined by D, as explained in [Vi-1].

Furthermore, $tor(H_{\mathcal{A}_2}^{i,j}(G)) = tor(H_{a,b}(D))$ for $i = \ell - 1$.

We generalize Theorem 3.5 from an alternating diagram to any diagram by using the graph $G_{s_+}(D)$ which for alternating diagrams is a Tait graph. The proof follows exactly the same line as that of Theorem 3.5 given in [H-P-R]. Below we use notation from Theorem 3.5.

Theorem 3.6. *Let D be the diagram of an unoriented framed link and $G = G_{s_+}(D)$ its associated graph. Then:*

(i) *For all $i < \ell - 1$, we have*

$$H_{\mathcal{A}_2}^{i,j}(G) \cong H_{a,b}(D)$$

(ii) *For $i = \ell - 1$ we have $tor(H_{\mathcal{A}_2}^{i,j}(G)) = tor(H_{a,b}(D))$.*

If girth $\ell(G_{s_+}(D)) > 2$ we say that D is strongly $+$-adequate. From the main result of this section (Theorem 3.1) and Theorem 3.6 we get the following generalization of Theorem 3.4:

Corollary 3.7.   (i) *Assume that D is a $+$-adequate diagram, then*

$$tor(H_{N-2,N+2|s_+|-4}(D)) = \mathbb{Z}_2^{p_0(G_{s_+}(D)) - p_0^{bi}(G_{s_+}(D))}.$$

(ii) *Assume that D is a strongly $+$-adequate diagram. Then*

$$H_{N-2,N+2|s_+|-4}(D)) = \mathbb{Z}_2^{p_0 - p_0^{bi}} \oplus \mathbb{Z}^{p_1 - (p_0 - p_0^{bi})}.$$

(iii) *Assume that D is a $-$-adequate diagram, then*

$$tor(H_{-N+2,-N-2|s_-|+4}(D)) = \mathbb{Z}_2^{p_0(G_{s_-}(D)) - p_0^{bi}(G_{s_-}(D))}.$$

(iv) *Assume that D is a strongly $--$adequate diagram. Then*

$$H_{-N+2,-N-2|s_-|+4}(D)) = \mathbb{Z}_2^{p_0-p_0^{bi}} \oplus \mathbb{Z}^{p_1-(p_0-p_0^{bi})}.$$

We can associate to any Kauffman state s not only the graph $G_s(D)$ but also a surface, $F_s(D)$, such that $G_s(D)$ is the spine of $F_s(D)$ (we generalize in such a way Tait's black and white surfaces of checkerboard coloring and the Seifert surfaces of an oriented diagram; compare Footnote 4.). We can rephrase Corollary 3.7(i) to say that the torsion of $H_{N-2,N+2|s_+|-4}(D)$ is $\mathbb{Z}_2^n$ where n is the number of unoriented components of the surface $F_{s_+}(D)$.

We also speculate that there is a relation of $\mathcal{A}_m$ graph homology to $sl(m)$ Khovanov-Rozansky [K-R-1, K-R-2] homology and/or colored Jones homology of links [Kh-2].

To put Theorem 3.1 and Corollary 3.5 in perspective let us recall that very little is known about torsion in Khovanov homology of links. For a while it was thought that the only possible torsion is 2-torsion (i.e. $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, ...$). Then Bar-Natan announced that torus knots can have odd torsion (e.g. homology of the torus knot of type $(8,7)$ has $\mathbb{Z}_3, \mathbb{Z}_5$ and $\mathbb{Z}_7$ in its torsion [BN-3, BN-4][5]). For prime alternating links the only torsion found so far is $\mathbb{Z}_2$ torsion. A. Shumakovitch proved that there is only 2-torsion in Khovanov homology of alternating links and he conjectures that $\mathbb{Z}_4$-torsion is impossible[6]. Shumakovitch proved also that every alternating link which is not disjoint or connected sum of Hopf links or trivial links has $\mathbb{Z}_2$ torsion [Shu]. In [A-P] we found explicity $\mathbb{Z}_2$ torsion in many adequate links, recovering in particular, the result of Shumakovitch. In [H-P-R] we proved the result from [A-P] in the $\mathcal{A}_2$ graph cohomology setting. In particular we proved that a simple graph, which is not a forest has $\mathbb{Z}_2$ in $H_{\mathcal{A}_2}^{1,v-1}(G)$ if G has an odd cycle and it has $\mathbb{Z}_2$ in $H_{\mathcal{A}_2}^{2,v-2}(G)$ if G has and even cycle. In this section we have computed completely $H_{\mathcal{A}_2}^{1,v-1}(G)$, showing, in particular, that if G is connected then the torsion part of $H_{\mathcal{A}_2}^{1,v-1}(G)$ is trivial if G is bipartite and it is $\mathbb{Z}_2$ otherwise (i.e. G has an odd cycle). In particular, the version of Shumakovitch conjecture for graphs holds for height one ($H_{\mathcal{A}_2}^{1,*}(G)$ has no $\mathbb{Z}_4$ in its torsion part).

4. COMPUTATION OF $H_{\mathcal{A}_3}^{1,2v-3}(G)$

The main result of this section describes the cohomology (and homology) at degree $2v - 3$. The result is described using certain cell complex built from a graph G: $X_{\Delta,4}$ is the cell complex obtained from G by adding 2-cells along 4-cycles in G, identifying all vertices of G and finally adding 2-cells

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[5]For example the 22nd homology at degree 73 is equal to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$.

[6]We use p-torsion to describe elements of order p^i for a prime p while $\mathbb{Z}_n$-torsion denotes an element of order n.

along expressions $2\vec{e}_3 - \vec{e}_2 - \vec{e}_1$ for any 3-cycle in G – two 2-cells added per every 3-cycle[7] (see Figure 4.9).

Theorem 4.1. *For an arbitrary simple graph G with v vertices and chromatic homology over algebra $\mathcal{A}_3$ the following is true:*

(0) $H_{\mathcal{A}_3}^{0,2v-3}(G)$ *is a free abelian group isomorphic to* $H^1(X_{\Delta,4}, \mathbb{Z}) \oplus \mathbb{Z}^{t_0 + \frac{d_2}{2} + d_{\geqslant 3}}$, *where t_0 is the number of unoriented triplets of vertices not connected by any edge, d_2 is the number of ordered pairs of vertices of distance two and $d_{\geqslant 3}$ is the number of ordered pairs of vertices of distance at least three.*

(1) $H_{\mathcal{A}_3}^{1,2v-3}(G) = H^2(X_{\Delta,4}, \mathbb{Z}) \oplus \mathbb{Z}^{t_2 - \frac{d_2}{2} - sq(G)}$, *where[8] t_2 is the number of unoriented triplets of vertices connected by exactly two edges (we call such a configuration, a joint), and $sq(G)$ denotes the number of squares (i.e. 4-cycles) in G.*

Below is the reformulation of Theorem 4.1 in the language of homology which will be used to prove our main result of this section.

Theorem 4.2. *For an arbitrary simple graph G with v vertices and chromatic homology over algebra $\mathcal{A}_3$ the following is true:*

(0) $H_{0,2v-3}^{\mathcal{A}_3}(G) = H_1(X_{\Delta,4}, \mathbb{Z}) \oplus \mathbb{Z}^{t_0 + \frac{d_2}{2} + d_{\geqslant 3}}$,

(1) $H_{1,2v-3}^{\mathcal{A}_3}(G)$ *is a free abelian group isomorphic to*
$H_2(X_{\Delta,4}, \mathbb{Z}) \oplus \mathbb{Z}^{t_2 - \frac{d_2}{2} - sq(G)}$.

Since the proof of Theorem 4.2 requires a lot of technical details we will first give a brief outline containing main ideas and then the proof itself. As we have already mentioned in Section 2.5 calculating homology instead of cohomology enables us to establish straightforward connections to the homology of a cell complex corresponding to our graph. In order to get more information about $H_{0,2v-3}^{\mathcal{A}_3}(G)$ and $H_{1,2v-3}^{\mathcal{A}_3}(G)$ we will calculate homology with coefficients in $\mathbb{Z}_3$ and $\mathbb{Z}[\frac{1}{3}]$ (i.e. the localization on the multiplicative set generated by 3). In some special cases we will be able to distinguish $\mathbb{Z}_{3^i}$-torsion for an arbitrary i by computing $3H_{0,2v-3}^{\mathcal{A}_3}(G)$.

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[7]To have uniquely defined cell complex we would have to add three 3-cells, but because then attachments would be linearly dependent: $(2\vec{e}_3 - \vec{e}_2 - \vec{e}_1) + (2\vec{e}_3 - \vec{e}_2 - \vec{e}_1) + (2\vec{e}_3 - \vec{e}_2 - \vec{e}_1) = 0$, thus from the point of view of $H^1(X_{\Delta,4}, \mathbb{Z})$ or $H_1(X_{\Delta,4}, \mathbb{Z})$, it suffices to add two 2-cells. The different choice of two 2-cells for any 3-cycle of G may however change the fundamental group of $X_{\Delta,4}$. This can be repaired by choosing relations of the form $\vec{e}_{i+1} - \vec{e}_i + \vec{e}_{i+1} - \vec{e}_{i-1}$, i=1,2,3, or in multiplicative notation $\vec{e}_i \vec{e}_{i-1}^{-1} \vec{e}_i \vec{e}_{i+1}^{-1}$. With this choice we have the identity $(\vec{e}_3 \vec{e}_2^{-1} \vec{e}_3 \vec{e}_1^{-1})(\vec{e}_1 \vec{e}_3^{-1} \vec{e}_1 \vec{e}_2^{-1})(\vec{e}_2 \vec{e}_1^{-1} \vec{e}_2 \vec{e}_3^{-1}) = 1$, thus $\pi_1(X_{\Delta,4})$ would not depend on the choice of two 2-cell attachments out of three possibilities. In the case of $G = P_3$, a triangle, we get $\pi_1((P_3)_{\Delta,4}) = \mathbb{Z}_3 * \mathbb{Z}$.

[8]$t_2 - \frac{d_2}{2} - sq(G)$ can be a negative number so formally it would be better to brake the formula into torsion and free part, that is: $tor H_{\mathcal{A}_3}^{1,2v-3}(G) = tor H^2(X_{\Delta,4}, \mathbb{Z})$ and $rank H_{\mathcal{A}_3}^{1,2v-3}(G) = rank H^2(X_{\Delta,4}, \mathbb{Z}) + t_2 - \frac{d_2}{2} - sq(G)$. Compare Remark 4.4.

Recall that $C_0(G) \cong \mathcal{A}_m^{\otimes v}$ and because $\mathcal{A}_m$ is a free abelian group with basis $1, x, x^2, ..., x^{m-1}$, thus $C_0^{\mathcal{A}_m}$ has basis of v-tuples. In particular, $C_{0,(m-1)(v-2)+1}^{\mathcal{A}_m}(G)$ has basis of v-tuples $(x^{a_1}, x^{a_2}, ..., x^{a_v})$ with $0 \le a_i < m$ and $\sum_{i=1}^m a_i = (m-1)(v-2) + 1$. Similarly, $C_1(G) \cong \oplus_E \mathcal{A}_m^{\otimes v-1}$ where E will be used to denote set of edges in our graph G and its cardinality, as long as it causes no confusion. Because G is a simple graph $(\ell(G) \ge 2)$ therefore for $|s| = 2$ the graph $[G : S]$ has $k(s) = v - 2$ components and thus $C_{2,(m-1)(v-2)+1}^{\mathcal{A}_m} = 0$ and $H_1^{\mathcal{A}_m}(G) = \ker(d_1)$. We work in this section with $m = 3$ and grading $2v - 3$. In this grading our chain complex is as follows:

$$0 \leftarrow C_{0,2v-3}(G) \xleftarrow{d_1} C_{1,2v-3}(G) \xleftarrow{d_2} 0$$

The chain groups in this grading can be easily expressed in terms of number of vertices v and number of edges E of our graph:

$C_{0,2v-3}^{\mathcal{A}_3}(G)$ has two essentially different types of generators:

- unordered triples $(v_i, v_j, v_k) = (x, x, x)$; vertices v_i, v_j, v_k having weights equal to x; remaining vertices having weights x^2
- ordered pairs $(v_i, v_j) = (1, x)$; weights of v_i and v_j being equal to 1 and x respectively; remaining vertices having weights x^2

In particular $C_{0,2v-3}^{\mathcal{A}_3}(G) \cong \mathbb{Z}^{\binom{v}{3}+v(v-1)}$.

Similarly, $C_{1,2v-3}^{\mathcal{A}_3}$ has two types of generators

- (e_i) with weight x and all isolated vertices have weights x^2
- (e_i, v_j) where weight of the edge is x^2 and v_j is the only vertex with weight x, the rest have weight x^2.

In particular, $C_{1,2v-3}^{\mathcal{A}_3}(G) \cong \mathbb{Z}^{E(v-1)}$. Recall that boundary map in cohomology uses multiplication in algebra, therefore differential in homology is its dual comultiplication. In particular, we consider all possible splits of components (in the case of $C_{1,2v-3}$ edge is mapped to its endpoints) and on the algebraic level this corresponds to all possible factorizations of a basic element x^k representing weight of the particular component being split into x^i and x^{k-i}.

Using the above notation for basis we describe decompositions of $C_{0,2v-3}$ as a direct sum of free abelian groups:

$$C_{0,2v-3}(G) = C_{0,2v-3}^{(t_0)} \oplus C_{0,2v-3}^{(t_1)} \oplus C_{0,2v-3}^{(t_2)} \oplus C_{0,2v-3}^{(t_3)} \oplus C_{0,2v-3}^{(d_1)} \oplus C_{0,2v-3}^{(d_2)} \oplus C_{0,2v-3}^{(d_3)}$$

where the summands are defined as follows:

- $C_{0,2v-3}^{(t_0)}$ is freely spanned by all triples (v_i, v_j, v_k) with no connections between vertices v_i, v_j, v_k in the graph G
- $C_{0,2v-3}^{(t_1)}$ is freely spanned by all triples (v_i, v_j, v_k) with exactly one of the edges between v_i, v_j, v_k present in the graph G

- $C^{(t_2)}_{0,2v-3}$ is freely spanned by all triples (v_i, v_j, v_k) with exactly 2 of edges between vertices v_i, v_j, v_k belonging to the graph G
- $C^{(t_3)}_{0,2v-3}$ is freely spanned by all triples (v_i, v_j, v_k) with all 3 edges between vertices v_i, v_j, v_k present in the graph
- $C^{(d_1)}_{0,2v-3}$ is freely spanned by all ordered pairs (v_i, v_j) with the distance $d(v_i, v_j) = 1$
- $C^{(d_2)}_{0,2v-3}$ is freely spanned by all ordered pairs (v_i, v_j) with the distance $d(v_i, v_j) = 2$
- $C^{(d_{\geq 3})}_{0,2v-3}$ is freely spanned by all ordered pairs (v_i, v_j) with the distance $d(v_i, v_j) \geq 3$

We use the convention that $t_0, t_1, t_2, t_3, d_1, d_2, d_{\geq 3}$ without parenthesis are the actual numbers and represent the ranks of those groups.

Now we explain the direct sum decomposition of $C_{1,2v-3}$:

$$C_{1,2v-3}(G) = C^{(E)}_{1,2v-3} \oplus C^{(t_1)}_{1,2v-3} \oplus C^{(2t_2)}_{1,2v-3} \oplus C^{(3t_3)}_{1,2v-3}$$

where the following groups are subgroups of $C_{1,2v-3}$, $e = \overline{v_i v_j}$ denotes and edge whose endpoints are vertices v_i, v_j:

- $C^{(E)}_{1,2v-3}$ is freely generated by states $(e) = (x)$.
- $C^{(t_1)}_{1,2v-3}$ is freely generated by states $(e, v_k) = (x^2, x)$ where e is the only edge between vertices v_i, v_j, v_k in the graph G.
- $C^{(2t_2)}_{1,2v-3}$ is freely generated by states $(e, v_k) = (x^2, x)$ where exactly two edges between v_i, v_j, v_k are present in G.
- $C^{(3t_3)}_{1,2v-3}$ is freely generated by all states $(e, v_k) = (x^2, x)$ where all three edges between v_i, v_j, v_k are present in G.

Now we can use these direct sum decomposition to analyze our chain complex and extract its parts that give free part of homology..

Step 1. According to the definition of the differential:

$$\operatorname{im} d_1 \subset \widehat{C^{(t_0)}_{0,2v-3}} \oplus C^{(t_1)}_{0,2v-3} \oplus C^{(t_2)}_{0,2v-3} \oplus C^{(t_3)}_{0,2v-3} \oplus$$

$$\oplus C^{(d_1)}_{0,2v-3} \oplus C^{(d_2)}_{0,2v-3} \oplus C^{(d_3)}_{0,2v-3},$$

where $\widehat{C}$ denotes the summand which is deleted from the sum.

Therefore $C^{(t_0)}_{0,2v-3}(G)$ contributes $\mathbb{Z}^{t_0}$ to homology $H^{\mathcal{A}_3}_{0,2v-3}(G)$.

Step 2. Now we will show that $d_1(C^{(E)}_{1,2v-3}(G)) \subset C^{(d_1)}_{0,2v-3}(G)$. As before, let $e = \overline{v_i v_j}$. Then:

$$(1) \qquad\qquad d_1(\overline{v_i v_j}) = (1, x) + (x, 1) = \overrightarrow{e} + \overleftarrow{e}$$

In other words, the following equivalent relation holds for every edge e in E:

$$\overrightarrow{e} = -\overleftarrow{e}. \tag{2}$$

We use this relation to reduce the size of chain complex by $\frac{d_1}{2} = E$:

$$C^{(d_1)}_{0,2v-3}(G)/d_1(C^{(E)}_{1,2v-3}(G)) = C^{(E)}_{0,2v-3}(G) \cong \mathbb{Z}^E \cong \mathbb{Z}^{2E}/(\overrightarrow{e} = -\overleftarrow{e}). \tag{3}$$

Step 3. If exactly two vertices, say v_i, v_j, are connected by an edge in the graph G, then we have the isomorphism between $C^{(t_1)}_{0,2v-3}(G)$ and $C^{(t_1)}_{1,2v-3}$. More precisely:

$$d_1(\overline{v_i v_j}) = (x,x,x) + (1,x^2,x) + (x^2,1,x) \tag{4}$$

This relation is used to present (x,x,x) states by other elements in a unique way.

Step 4. If exactly two edges are present in the graph G we are analyzing:

$$d_1(C^{(2t_2)}_{1,2v-3}) \subset C^{(t_2)}_{0,2v-3}(G) \oplus C^{(E)}_{0,2v-3}(G) \oplus C^{(d_2)}_{0,2v-3}(G).$$

Without loss of generality, let those two edges be denoted by $\overline{v_i v_j}, \overline{v_j v_k}$; then we have:

$$d_1(\overline{v_i v_j}) = (x,x,x) + (1,x^2,x) + (x^2,1,x)$$
$$d_1(\overline{v_i v_k}) = (x,x,x) + (1,x,x^2) + (x^2,x,1)$$

Thus we can eliminate states (x,x,x) and be left with relation

$$(x,x^2,1) - (1,x^2,x) = (x^2,1,x) - (x,1,x^2) = \overrightarrow{e_1} - \overleftarrow{e_2} \tag{5}$$

This relation allows us to eliminate half of elements of $C^{d_2}_{0,2v-3}(G)$ and the other half contributes $\mathbb{Z}^{d_2/2}$ to homology $H^{\mathcal{A}_3}_{0,2v-3}(G)$. If there is another vertex v_l from graph G such that $\overline{v_i v_l}, \overline{v_k v_l} \in E$ then, in a similar way we have the relation:

$$(x,x^2,1) - (1,x^2,x) = (x^2,x,1) - (x,x^2,1) = \overrightarrow{e_3} - \overleftarrow{e_4} \tag{6}$$

Combining the above two relations using the common expression on the left side of each equation we get:

$$\overrightarrow{e_1} - \overleftarrow{e_2} + \overrightarrow{e_3} - \overleftarrow{e_4} = 0 \tag{7}$$

or equivalently:

$$\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3} + \overrightarrow{e_4} = 0 \tag{8}$$

This relation will be used later in the proof of Main Lemma.
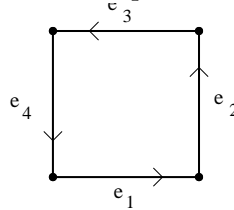

Figure 4.7

Step 5. Finally, if all three edges are present in the graph G (forming a triangle in G) then we have:

$$d_1(C_{1,2v-3}^{(3t_3)}) \subset C_{0,2v-3}^{(t_3)}(G) \oplus C_{0,2v-3}^{(E)}(G)$$

and as before we obtain following relations (for simplicity, we denote the three edges of the triangle by e_i, e_j, e_k):

$$(9) \quad d_1(\overline{v_i v_j}) = (x, x, x) + (1, x^2, x) + (x^2, 1, x) = (x, x, x) + \overleftarrow{e_k} + \overrightarrow{e_j}$$

$$(10) \quad d_1(\overline{v_j v_k}) = (x, x, x) + (x, 1, x^2) + (x, x^2, 1) = (x, x, x) + \overleftarrow{e_i} + \overrightarrow{e_k}$$

$$(11) \quad d_1(\overline{v_i v_k}) = (x, x, x) + (1, x, x^2) + (x^2, x, 1) = (x, x, x) + \overrightarrow{e_i} + \overleftarrow{e_j}$$

Therefore we can reduce the size of the chain group $C_{0,2v-3}$ by eliminating (x, x, x) states. After the elimination we are left with two relations:

$$(12) \qquad\qquad \vec{e}_i - \vec{e}_j = \vec{e}_j - \vec{e}_k = \vec{e}_k - \vec{e}_i$$

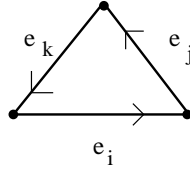where edges of the triangle v_i, v_j, v_k are coherently oriented (Fig. 4.8).

Figure 4.8:

Step 6. To formulate Main Lemma we need one more definition. Let $I_{\Delta,4}$ denote the subgroup of $\mathbb{Z}^E$ generated by two types of elements:

(1) Every square in our graph corresponds to one generator of $I_{\Delta,4}$:

$$u_4 := \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3} + \overrightarrow{e_4}$$

(2) Every triangle from graph G contributes three generators to $I_{\Delta,4}$:

$$u_1 := (\overrightarrow{e_1} - \overrightarrow{e_2}) - (\overrightarrow{e_2} - \overrightarrow{e_3})$$

$$u_2 := (\overrightarrow{e_2} - \overrightarrow{e_3}) - (\overrightarrow{e_3} - \overrightarrow{e_1})$$

$$u_3 := (\overrightarrow{e_3} - \overrightarrow{e_1}) - (\overrightarrow{e_1} - \overrightarrow{e_2})$$

Note that these generators are linearly dependent since
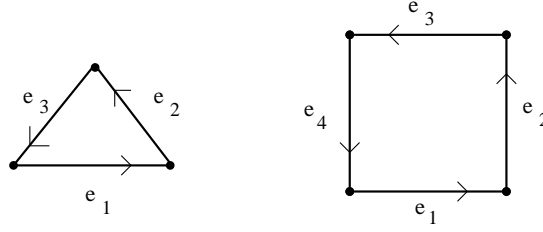$u_1 + u_2 + u_3 = 0$.

Figure 4.9:
With this notation we can now summarize our analysis in Main Lemma.

Lemma 4.3 (Main Lemma).

$$H_{0,2v-3}^{\mathcal{A}_3}(G) = \mathbb{Z}^E/I_{\Delta,4} \oplus \mathbb{Z}^{t_0+\frac{d_2}{2}+d_{\geqslant 3}}.$$

Proof. First, $\mathbb{Z}^{t_0+\frac{d_2}{2}+d_{\geqslant 3}}$ is the free summand of $H_{0,2v-3}^{\mathcal{A}_3}(G)$ by Steps 1, 2, and by the fact that $im(d_1(C_{1,2v-3}(G)))$ belongs to the direct sum of other then $\mathbb{Z}^{d_{\geqslant 3}}$ summands of $C_{0,2v-3}^{\mathcal{A}_3}(G)$. Recall that all generators of $C_{0,2v-3}$ are of the form (x,x,x) or $(1,x)$ and in the case endpoints of an edge e_i have weights 1 and x we write $(1,x) = \overrightarrow{e_i}$ or $(x,1) = \overleftarrow{e_i}$. In Steps 3, 4 and 5 we described how to eliminate generators of type (x,x,x). In Step 2 we found relation $\overrightarrow{e} = -\overleftarrow{e}$. In Step 4 we also eliminated half of relations of the form $(1,x)$ when the distance between points is equal to two. The other half contributes $\mathbb{Z}^{\frac{d_2}{2}}$ to $H_{0,2v-3}^{\mathcal{A}_3}(G)$. So finally we are left with E generators of type $\overrightarrow{e}$ spanning $\mathbb{Z}^E$. Still, we have relations coming from squares and triangles (Steps 4 and 5). These are exactly the relations generating $I_{\Delta,4}$.

Hence, $H_{0,2v-3}^{\mathcal{A}_3}(G) = \mathbb{Z}^E/I_{\Delta,4} \oplus Z^{t_0+\frac{d_2}{2}+d_{\geqslant 3}}$.   □

We are ready now to prove Theorems 4.1 and 4.2.

Proof. We start from Theorem 4.2(0). Observe that $H_1(X_{\Delta,4};\mathbb{Z})$ is equal to $\mathbb{Z}^E/I_{\Delta,4}$ because the cell complex $X_{\Delta,4}$ has as one skeleton the graph G with all vertices identified so 1-cycles have E as a basis. Furthermore, 2-cells of $X_{\Delta,4}$ were chosen in such a way that their boundaries generate the subgroup $I_{\Delta,4}$. Therefore Theorem 4.2(0) follows from Main Lemma (Lemma 4.3). Theorem 4.2(1) follows from the fact that $H_{1,2v-3}^{\mathcal{A}_3}(G) = ker(d_1 : C_{1,2v-3} \to C_{0,2v-3})$ so it is a free abelian group of the rank equal to

$$rank\, C_{1,2v-3} - rank\, C_{0,2v-3} + rank\, H_{0,2v-3}^{\mathcal{A}_3}(G)$$

$$= E(v-1) - \binom{v}{3} - v(v-1) + rank\, H_{0,2v-3}^{\mathcal{A}_3}(G).$$

Furthermore

$$rank\, H_{0,2v-3}^{\mathcal{A}_3}(G) = rank\, H_1(X_{\Delta,4}(G)) + t_0 + \frac{d_2}{2} + d_{\geqslant 3}$$

and

$$rank\, H_1(X_{\Delta,4}(G)) = rank\, H_2(X_{\Delta,4}(G)) + rank\, H_0(X_{\Delta,4}(G)) + E - 2t_3 - sq(G) - 1$$

$$= rank\, H_2(X_{\Delta,4}(G)) + E - 2t_3 - sq(G).$$

Combining these together we get:

$rank\, H_{1,2v-3}^{\mathcal{A}_3}(G) =$

$rank\, H_2(X_{\Delta,4}(G)) + E(v-1) - \binom{v}{3} - v(v-1) + t_0 + \frac{d_2}{2} + d_{\geqslant 3} + E - 2t_3 - sq(G)$

$= rank\, H_2(X_{\Delta,4}(G)) + t_2 - \frac{d_2}{2} - sq(G) +$

$(-t_2 + E(v-1) - \binom{v}{3} - v(v-1) + t_0 + d_2 + d_{\geqslant 3} + E - 2t_3)$

$= rank\, H_2(X_{\Delta,4}(G)) + t_2 - \frac{d_2}{2} - sq(G),$

as required. The last equality follows from the identities:

$$v(v-1) = d_1 + d_2 + d_{\geqslant 3} = 2E + d_2 + d_{\geqslant 3},$$

$$E(v-1) = E + t_1 + 2t_2 + 3t_3,$$

$$\binom{v}{3} = t_0 + t_1 + t_2 + t_3,$$

and therefore

$$-t_2 + E(v-1) - \binom{v}{3} - v(v-1) + t_0 + d_2 + d_{\geqslant 3} + E - 2t_3 = 0.$$

Theorem 4.1 follows from Theorem 4.2 by applying Proposition 2.9.   □

Remark 4.4. *If G has a 4-cycle with a diagonal (Figure 4.10) then the relation $0 = \vec{e_1} + \vec{e_2} + \vec{e_3} + \vec{e_4}$ yielded by this 4-cycle can be obtained from relations obtained from it follows from triangle relations associated to triangles dividing the 4-cycle. Namely, triangles give relations: $2\vec{e} = \vec{e_1} + \vec{e_2}$ and $-2\vec{e} = \vec{e_3} + \vec{e_4}$, whose sum is exactly the relation from the 4-cycle. Consequently, it may be useful to consider the cell complex $X_{\Delta,4'} \subset X_{\Delta,4}$ in which we glue 2-cells along 4-cycles only in the case in which the 4-cycle has no diagonal. We observe that $H_1(X_{\Delta,4'}, \mathbb{Z}) = H_1(X_{\Delta,4}, \mathbb{Z})$ and $H_2(X_{\Delta,4'}, \mathbb{Z}) \oplus \mathbb{Z}^{sq'(G)} = H_1(X_{\Delta,4}, \mathbb{Z})$, where $sq'(G)$ denotes the number of 4-cycles in G which have a diagonal. In this notation Theorem 4.1(1) has the form:*

$$H_{\mathcal{A}_3}^{1,2v-3}(G) = H^2(X_{\Delta,4'}, \mathbb{Z}) \oplus \mathbb{Z}^{t_2 - \frac{d_2}{2} - sq(G) + sq'(G)}.$$

In Lemma 4.7 we will consider graphs in which every 4-cycle has a diagonal, called square cordial (e.g. complete graphs or wheels). In this case $X_{\Delta,4'} = X_\Delta$ (no 2-cell is attached to a 4-cycle) which allows simpler formulation of Theorems 4.1 and 4.2.
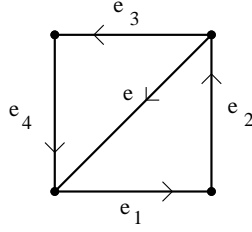
Figure 4.10:

The case of square cordial graphs will be revisited in Lemma 4.7 and Corollary 4.8.

Computing homology of the cell complex $X_{\Delta,4}(G)$ is cumbersome, however we can recover a substantial part of it by considering three simpler complexes (compare Footnote 11):

(i) $X_{(3),4}$ obtained from G by identifying edges of every triangle in a coherent way (i.e. $\vec{e}_i = \vec{e}_j = \vec{e}_k$ for a triangle oriented as in Figure 4.8), then adding 2-cells along every 4-cycle in G and finally identifying all vertices of G.

(ii) $\hat{X}_{3,4}$ obtained from G by adding 2-cells along every 3- and 4-cycle of G.

(iii) $X_{3,4}$ obtained from $\hat{X}_{3,4}$ by identifying all vertices of G.

Observe that $H_2(X_{3,4}) = H_2(\hat{X}_{3,4}), H_1(X_{3,4}) = H_1(\hat{X}_{3,4}) \oplus \mathbb{Z}^{v-1}$ and $H_1(X) \otimes R = H_1(X, R)$ according to the Universal Coefficient Theorem (e.g. [Hat]), (compare also proof of Proposition 4.5).

Proposition 4.5.

(a) $H_1(X_{\Delta,4}) \otimes \mathbb{Z}_3 = H_1(X_{\Delta,4}, \mathbb{Z}_3) = H_1(X_{3,4}, \mathbb{Z}_3) = H_1(X_{3,4}) \otimes \mathbb{Z}_3$.

(b) $H_1(X_{\Delta,4}) \otimes \mathbb{Z}[\frac{1}{3}] = H_1(X_{(3),4}) \otimes \mathbb{Z}[\frac{1}{3}] = H_1(X_{(3),4}, \mathbb{Z}[\frac{1}{3}])$.

Proof. We know that $H_1(X_{\Delta,4}, \mathbb{Z}) = \mathbb{Z}^E/I_{\Delta,4}$ so we will consider the following short exact sequence:

$$C : 0 \to I_{\Delta,4} \to \mathbb{Z}^E \to \mathbb{Z}^E/I_{\Delta,4} \to 0$$

From right exactness of tensor product we get that the sequence:

$$I_{\Delta,4} \otimes R \to \mathbb{Z}^E \otimes R \to \mathbb{Z}^E/I_{\Delta,4} \otimes R \to 0$$

is exact for every ring R. Hence

$$(\mathbb{Z}^E/I_{\Delta,4}) \otimes R = (\mathbb{Z}^E \otimes R)/(I_{\Delta,4} \otimes R) = R^E/I_{\Delta,4}^R.$$

where $I_{\Delta,4}^R$ is a submodule of R^E constructed in the same way as $I_{\Delta,4} = I_{\Delta,4}^{\mathbb{Z}}$.

In particular, $H_1(X_{\Delta,4}) \otimes R = H_1(X_{\Delta,4}, R)$. To prove Proposition 4.5 we consider cases $R = \mathbb{Z}_3$ and $R = \mathbb{Z}[\frac{1}{3}]$ separately.

(a) Let $R = \mathbb{Z}_3$. Then $H_1(X_{\Delta,4}, \mathbb{Z}_3) = \mathbb{Z}_3^E/I_{\Delta,4}^{\mathbb{Z}_3}$

To find $I_{\Delta,4}^{\mathbb{Z}_3}$ notice that in $\mathbb{Z}_3$ we have:

$$e_1 + e_3 - 2e_2 = e_1 + e_2 + e_3 - 3e_2 = e_1 + e_2 + e_3.$$

Similarly $e_2 + e_1 - 2e_2 = e_1 + e_2 + e_3 = e_3 + e_2 - 2e_1$. Thus $I_{\Delta,4}^{\mathbb{Z}_3}$ is generated by expressions coming only from 3 and 4-cycles:

$$\text{(13)} \qquad \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}$$

$$\text{(14)} \qquad \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3} + \overrightarrow{e_4}$$

In this way we obtain that $\mathbb{Z}_3^E / I_{\Delta,4}^{\mathbb{Z}_3}$ is exactly the first homology of the cell complex $X_{3,4}(G)$ with $\mathbb{Z}_3$ coefficients. That is, $\mathbb{Z}_3^E / I_{\Delta,4}^{\mathbb{Z}_3} = H_1(X_{3,4}(G), \mathbb{Z}_3) = H_1(X_{3,4}(G), \mathbb{Z}) \otimes \mathbb{Z}_3$.

(b) Let $R = \mathbb{Z}[\frac{1}{3}]$. Then $H_1(X_{\Delta,4}, \mathbb{Z}[\frac{1}{3}]) = \mathbb{Z}[\frac{1}{3}]^E / I_{\Delta,4}^{\mathbb{Z}[\frac{1}{3}]}$

To find $I_{\Delta,4}^{\mathbb{Z}[\frac{1}{3}]}$ notice that because 3 is invertible in $\mathbb{Z}[\frac{1}{3}]$ therefore generators of $I_{\Delta,4}^{\mathbb{Z}[\frac{1}{3}]}$ coming from 3-cycles in G are $e_i - \frac{e_1+e_2+e_3}{3}$, $i = 1, 2, 3$ (they yield $e_1 = e_2 = e_3$ in the quotient). Other generators are coming from 4-cycles and as before are of the form $\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3} + \overrightarrow{e_4}$.

In this way we conclude that $\mathbb{Z}[\frac{1}{3}]^E / I_{\Delta,4}^{\mathbb{Z}[\frac{1}{3}]}$ is exactly the first homology of the cell complex $X_{(3),4}(G)$ with $\mathbb{Z}[\frac{1}{3}]$ coefficients. That is, $\mathbb{Z}[\frac{1}{3}]^E / I_{\Delta,4}^{\mathbb{Z}[\frac{1}{3}]} = H_1(X_{(3),4}(G), \mathbb{Z}[\frac{1}{3}]) = H_1(X_{(3),4}(G), \mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{3}]$.

$\square$

In order to utilize and generalize Proposition 4.5 recall that every finitely generated abelian group H can be decomposed uniquely:

$$H = \text{free}(H) \oplus \text{tor}(H) = \mathbb{Z}^a \oplus \text{tor}_3(H) \oplus \text{tor}_{(3)}(H)$$

where $\text{tor}_3(H)$ contains summands of the form $\mathbb{Z}_{3^i}$, $i \in \{1, 2, ...\}$ and $\text{tor}_{(3)}(H)$ contains summands of the form $\mathbb{Z}_n$, where $\gcd(n, 3) = 1$.

If we find $H_{(3)} \overset{def}{=} H \otimes \mathbb{Z}[\frac{1}{3}]$ we recover exactly $\text{free}(H) \oplus \text{tor}_{(3)}(H)$ as

$$H \otimes \mathbb{Z}[\frac{1}{3}] = (\mathbb{Z}[\frac{1}{3}])^a \oplus \text{tor}_{(3)}(H).$$

If $\text{tor}_3(H) = \mathbb{Z}_3^{a_1} \oplus \mathbb{Z}_{3^2}^{a_2} \oplus \ldots \oplus \mathbb{Z}_{3^j}^{a_j}$ then

$$H \otimes \mathbb{Z}_3 = \mathbb{Z}_3^a \oplus (\text{tor}_{(3)}(H) \otimes \mathbb{Z}_3) = \mathbb{Z}_3^a \oplus \mathbb{Z}_3^{a_1+a_2+...+a_j}.$$

Therefore $H_{(3)}$ and H_3 allows us to find $\mathbb{Z}^a$ and $\text{tor}_{(3)}(H)$ part as well as the sum of exponents $a_1 + a_2 + \ldots + a_j$ in $\text{tor}_3(H)$ part. Of course these is not sufficient to distinguish $\mathbb{Z}_{3^s}$ from $\mathbb{Z}_3$. Sometimes, however, we can get enough information of $3H = \mathbb{Z}^a \oplus (\mathbb{Z}_3^{a_2} \oplus \mathbb{Z}_{3^2}^{a_3} \oplus \ldots \oplus \mathbb{Z}_{3^{j-1}}^{a_j}) \oplus \text{tor}_{(3)}(H)$ to compute whole H. We illustrate it by first generalizing slightly Proposition 4.5 and then computing homology for an important class of graphs (square-cordial graphs), including the complete graphs, K_n and wheels, W_n (that is, cones over $(n-1)$-gons, Figure 8.3), in which cases $3H$ has no 3-torsion.

Proposition 4.6. *Let G be a simple graph and*

$$H_1(X_{\Delta,4}) = \mathbb{Z}^a \oplus tor_3(H_1(X_{\Delta,4})) \oplus tor_{(3)}(H_1(X_{\Delta,4}))$$

where $tor_3(H_1(X(\Delta,4)) = \mathbb{Z}_3^{a_1} \oplus \mathbb{Z}_{3^2}^{a_2} \oplus \ldots \oplus \mathbb{Z}_{3^j}^{a_j}$ then
(i) $rank H_1(X_{(3),4}, \mathbb{Z}[\frac{1}{3}]) = rank H_1(X_{(3),4}) = rank 3H_1(X_{\Delta,4}) = a$
(ii) $tor_{(3)}(H_1(X_{\Delta,4})) = tor_{(3)}(H_1(X_{(3),4}, \mathbb{Z}[\frac{1}{3}])) = tor_{(3)}(H_1(X_{(3),4})) = tor_{(3)}3(H_1(X_{\Delta,4}))$,
(iii) $dim(H_1(X_{(3),4}, \mathbb{Z}_3) = a + a_1 + a_2 + \ldots + a_j$
(iv) $tor_3 3H_1(X_{\Delta,4}) = \mathbb{Z}_3^{a_2} \oplus \mathbb{Z}_{3^2}^{a_3} \oplus \ldots \oplus \mathbb{Z}_{3^{j-1}}^{a_j}$. Furthermore $tor_3 3H_1(X_{\Delta,4})$
is the quotient of $tor_3 H_1(X_{(3),4})$.
In particular, if $H_1(X_{(3),4})$ has no 3-torsion then $H_1(X_\Delta,4)$ has no $\mathbb{Z}_9$-torsion and $H_1(X_{\Delta,4})$ is fully determined by $H_1(X_{(3),4})$ and $H_1(X_{3,4}, \mathbb{Z}_3)$.

Proof. Parts (i),(ii), (iii) are just the reformulation of Proposition 4.5. The first part of (iv) is obvious for any finitely generated group. The second part of (iv) follows from the fact that the equality $3\vec{e}_i = 3\vec{e}_j = 3\vec{e}_k$ holds for any edges of coherently oriented triangle (Figure 4.8) and edges $3e$ generate $3H_1(X_\Delta,4)$. $3H_1(X_{\Delta,4})$ can have more relations than $H_1(X_{(3),4})$ thus we have the epimorphism from $H_1(X_{(3),4})$ to $3H_1(X_{\Delta,4})$ sending e to $3e$ for any edge e. Because both groups have $\mathbb{Z}^a$ as a free part, therefore the 3-torsion part of $3H_1(X_{\Delta,4})$ is equal to a quotient of the 3-torsion part of $H_1(X_{(3),4})$. $\square$

A graph G is called square-cordial if every 4-cycle in G has a diagonal. In order to be able to formulate our result about homology of square-cordial graphs, we define a new graph denoted by G_Δ and called the graph of triangles of G. The graph G_Δ is obtained from G as follows: vertices of G_Δ are in bijection with 3-cycles of G. Two vertices of G_Δ are connected by an edge if corresponding triangles share an edge (we may assume that if the triangles share two edges than vertices are connected by two edges). We say that the component of the graph G_Δ is coherent if relations $\vec{e}_i = \vec{e}_j = \vec{e}_k$ for every triangle in the component (compare Figure 4.8) never lead to relation of type $\overrightarrow{e} = \overleftarrow{e}$ (i.e. $2\vec{e} = 0$).

Lemma 4.7. *Let G be a simple square-cordial graph, then*

$$H_1(X_\Delta) = \mathbb{Z}^{p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}_2^{p_0(G_\Delta) - p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}_3^{v-1+dim H_1(\hat{X}_3, \mathbb{Z}_3) - p_0(G_\Delta)^{coh}},$$

where $p_0(G_\Delta)^{coh}$ is the number of coherent components of G_Δ.
In particular $H_1(X_\Delta)$ has no $\mathbb{Z}_9$-torsion.

Proof. If a triangle graph G_Δ is connected, then clearly

$$H_1(X_{(3)}) = \begin{cases} \mathbb{Z} & \text{if } G_\Delta \text{ is coherent} \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

Thus for any simple square-cordial graph we have:

$$H_1(X_{(3)}) = \mathbb{Z}^{p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}_2^{p_0(G_\Delta) - p_0(G_\Delta)^{coh}}.$$

By Proposition 4.6, $H_1(X_\Delta) = \mathbb{Z}^{p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}_2^{p_0(G_\Delta)-p_0(G_\Delta)^{coh}} \oplus \text{tor}_3 H_1(X_\Delta)$, and $\text{tor}_3 H_1(X_\Delta)$ has no $\mathbb{Z}_9$ torsion.

Furthermore, we have $H_1(X_3) = H_1(\hat{X}_3) \oplus \mathbb{Z}^{v-1}$. Combining these arguments together and applying Proposition 4.6, we get
$\text{tor}_3 H_1(X_\Delta) = \mathbb{Z}_3^{v-1+dim H_1(\hat{X}_3,\mathbb{Z}_3)-p_0(G_\Delta)^{coh}}$ as needed. $\qquad\square$

Corollary 4.8. *If G is a simple square-cordial graph then*

$$H_{\mathcal{A}_3}^{1,2v-3}(G)$$
$$= \mathbb{Z}_2^{p_0(G_\Delta)-p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}_3^{v-1+dim H_1(\hat{X}_3,\mathbb{Z}_3)-p_0(G_\Delta)^{coh}} \oplus \mathbb{Z}^{p_0(G_\Delta)^{coh}+t_2+2t_3-E-\frac{d_2}{2}}.$$

Note that $H_1(X_\Delta(G)) = H_1(X_{\Delta,4}(G))$ by Remark 4.4 – we do not have to add 2-cells along 4-cycles in square-cordial graphs. Now proof follows directly from Theorem 4.2(0) and Lemma 4.7 .

Corollary 4.9. *For the complete graph with n vertices K_n, $n \geq 4$ we have*

$$H_{\mathcal{A}_3}^{1,2n-3}(K_n) = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{n-1} \oplus \mathbb{Z}^{\frac{n(n-1)(2n-7)}{6}}.$$

Proof. Since K_n is a square-cordial graph, G_Δ is connected. Consequently, $H_1(X_3) = 0$ (as every cycle is a boundary cycle) and for $n > 3$, G_Δ is not coherent. Therefore, by Lemma 4.7: $H_1(X_\Delta(K_n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{n-1}$. For the complete graph we have $t_0 = d_2 = d_{\geq 3} = 0$, so from Theorem 4.2(0) we get $H_{0,2n-3}^{\mathcal{A}_3}(K_n) = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{n-1}$. Because $H_{0,2n-3}^{\mathcal{A}_3}(K_n)$ is a torsion group, the chain map $C_{1,2v-3}^{\mathcal{A}_3} \to C_{0,2v-3}^{\mathcal{A}_3}$ is an epimorphism over Q. Thus

$$rank\, H_{1,2n-3}^{\mathcal{A}_3}(K_n) = rank\, C_{1,2v-3}^{\mathcal{A}_3} - rank\, C_{0,2v-3}^{\mathcal{A}_3}$$
$$= \frac{n(n-1)}{2}(n-1) - \left(\frac{n(n-1)(n-2)}{6} + n(n-1)\right) = \frac{n(n-1)(2n-7)}{6}.$$

Therefore $H_{1,2v-3}^{\mathcal{A}_3}(K_n) = \mathbb{Z}^{\frac{n(n-1)(2n-7)}{6}}$ and
$H_{\mathcal{A}_3}^{1,2n-3}(K_n) = tor H_{0,2n-3}^{\mathcal{A}_3}(K_n) \oplus H_{1,2n-3}^{\mathcal{A}_3}(K_n) = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{n-1} \oplus \mathbb{Z}^{\frac{n(n-1)(2n-7)}{6}}$.
The proof of Corollary is completed[9]. $\qquad\square$

Corollary 4.10. *If a simple graph G contains a triangle then $H_{\mathcal{A}_3}^{1,2v(G)-3}(G)$ contains $\mathbb{Z}_3$.*

Proof. If G has an oriented triangle with edges e_1, e_2, e_3 we have the following relation in $H_{0,2v-3}^{\mathcal{A}_3}(G)$:

$$3(e_1 - e_2) = 3(e_2 - e_3) = 3(e_3 - e_1) = 0.$$

We need to prove that $e_1 - e_2$ is not 0 in homology. Assume that $e_1 - e_2 = 0$. As G embeds in the complete graph, e_1, e_2 are also elements of $H_{0,2v-3}^{\mathcal{A}_3}(K_v)$

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[9]For the triangle K_3 we are getting $H_{0,3}^{\mathcal{A}_3}(K_3) = \mathbb{Z} \oplus \mathbb{Z}_3$, $H_{1,3}^{\mathcal{A}_3}(K_3) = 0$ and $H_{\mathcal{A}_3}^{1,3}(K_3) = \mathbb{Z}_3$ which do not agree with the formula from Corollary 4.9 because the graph G_Δ is coherent.

and for K_v we have more relations then for G, thus the equation $e_1 - e_2 = 0$ is valid also in $H_{0,2v-3}^{\mathcal{A}_3}(K_v)$. By symmetry of the complete graph we get that all edges are equal in homology group $H_{0,2v-3}^{\mathcal{A}_3}(K_v)$. Thus the homology cannot contain $\mathbb{Z}_3$. This contradicts Corollary 4.10 and thus $H_{0,2v-3}^{\mathcal{A}_3}(G)$ contains $\mathbb{Z}_3$ torsion. $\hfill\square$

Corollary 4.11. *Let W_k denote the wheel, that is, the graph which is a cone over $(k-1)$-gon (Figure 8.3). Then*

$$H_{\mathcal{A}_3}^{1,2n-3}(W_n) = \begin{cases} \mathbb{Z}_3^{n-2} \oplus \mathbb{Z}^n, & \text{if } n \text{ odd}; \\ \mathbb{Z}_3^{n-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{n-1}, & \text{if } n \text{ even}. \end{cases}$$

Proof. Corollary 4.11 follows from Lemma 4.7 by observing that the wheel W_n is a square cordial-graph with $H_1(X_3) = 0$ and $(W_n)_\Delta$ is connected and coherent if and only if n is odd. $\hfill\square$

In the next section we describe initially unexpected examples of torsion in cohomology, finding in particular that for any k there is a graph G such that $H_{\mathcal{A}_3}^{1,2v-3}(G)$ contains Z_k. Graphs we consider are not square-diagonal; nevertheless Lemma 4.7 is very useful in the analysis.

5. The family of G_n graphs

The following graph, denoted by G_4, has $\mathbb{Z}_5$ in cohomology, that is, $H_{\mathcal{A}_3}^{1,37}(G_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{19} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}^{25}$.
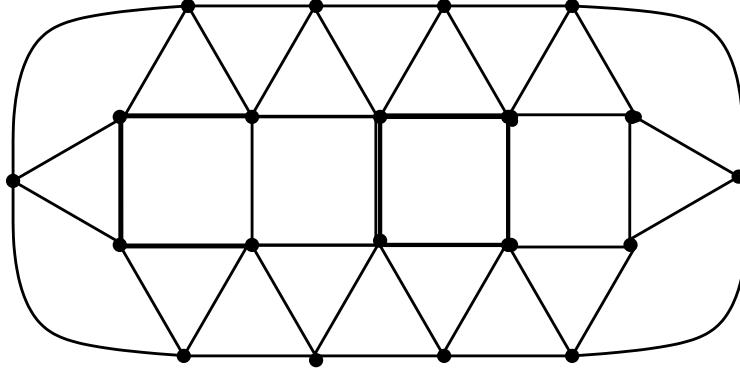

Figure 5.1

Initially we were surprised to see torsion different from $\mathbb{Z}_3$ or $\mathbb{Z}_2$. The reason was that A. Shumakovitch has conjectured that for alternating links the torsion in Khovanov homology can have only elements of order 2. We have conjectured analogously that for the algebra $\mathcal{A}_2$ the torsion part of $H_{\mathcal{A}_2}^{*,*}(G)$ can have only elements of order 2. This is still an open problem, and motivated by this we thought that for $\mathcal{A}_3$ graph homology torsion will be rather limited. However, after computing $H_{A_3}^{1,5}(K_4) = \mathbb{Z}_3^2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}^2$ we showed

that torsion can have elements of different order. Furthermore, we analyzed the family of plane graphs G_1, G_2,...,G_k (see Figures 5.1, 5.2) and we had a strong indication that for any n there is a graph with torsion $\mathbb{Z}_n$. In particular, $H_{\mathcal{A}_3}^{1,13}(G_1) = \mathbb{Z}_4 \oplus \mathbb{Z}_3^7 \oplus \mathbb{Z}^{12}$, $H_{\mathcal{A}_3}^{1,21}(G_2) = \mathbb{Z}_{18} \oplus \mathbb{Z}_3^{10} \oplus \mathbb{Z}^{15}$, $H_{\mathcal{A}_3}^{1,29}(G_3) = \mathbb{Z}_8 \oplus \mathbb{Z}_3^{15} \oplus \mathbb{Z}^{20}$, $H_{\mathcal{A}_3}^{1,37}(G_4) = \mathbb{Z}_{10} \oplus \mathbb{Z}_3^{19} \oplus \mathbb{Z}^{25}$, $H_{\mathcal{A}_3}^{1,45}(G_5) = \mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3^{22} \oplus \mathbb{Z}^{30}$, $H_{\mathcal{A}_3}^{1,53}(G_6) = \mathbb{Z}_{14} \oplus \mathbb{Z}_3^{27} \oplus \mathbb{Z}^{35}$, and $H_{\mathcal{A}_3}^{1,61}(G_7) = \mathbb{Z}_{16} \oplus \mathbb{Z}_3^{31} \oplus \mathbb{Z}^{40}$.

Graphs G_n are not square-diagonal so we cannot use Lemma 4.7 directly, however combining it with the Main Lemma we have proved that $H_{\mathcal{A}_3}^{1,8(k+1)-3}(G_k)$ contains $\mathbb{Z}_{6k+6}$ or more generally:

Corollary 5.1. *For $k > 1$ we have:*

$$H_{\mathcal{A}_3}^{1,8(k+1)-3}(G_k) = \mathbb{Z}_{6k+6} \oplus \mathbb{Z}_3^{4k+2} \oplus \mathbb{Z}^{5k+5}$$

Proof. Let G_k' be the graph obtained from G_k by deleting all edges which are not on 3-cycles (say $f_1, f_2, ..., f_{k-1}$). G_k' is a square-cordial graph with the triangle graph $(G_k')_\Delta$ connected and coherent. Therefore from Lemma 4.7 it follows that $H_1(X_\Delta(G_k')) = Z^{E(G_k')}/I_\Delta = Z \oplus \mathbb{Z}_3^{4k+3}$. Furthermore, $H_1(X_{\Delta,4}(G)) = \mathbb{Z}^{E(G)}/I_{\Delta,4} = \mathbb{Z}^{E(G_k')}/I_{\Delta,4} \cup R$, where R is the relation in which the sum of the edges along rectangle composed of squares is equal to 0. Combining this with previous relations we get the relation $(2k+2)e + h_\Delta$, where e is a generator of $\mathbb{Z}$ and h_Δ belongs to $\mathbb{Z}_3^{4k+3}$. The 3-torsion element h_Δ cannot be equal to zero because

$$(\mathbb{Z}^{E(G)}/I_{\Delta,4}) \otimes \mathbb{Z}_3 = H_1(X_{3,4}(G), \mathbb{Z}_3) = \mathbb{Z}_3^{4k+3} \oplus H_1(\hat{X}_{3,4}(G), \mathbb{Z}_3) = \mathbb{Z}_3^{4k+3}.$$

Therefore $H_1(X_{\Delta,4}(G)) = \mathbb{Z}_{6k+6} \oplus \mathbb{Z}_3^{4k+2}$. To find the free part of $H_{\mathcal{A}_3}^{1,8(k+1)-3}(G_k)$ we use the standard tricks using Euler characteristic and duality (homology – cohomology). $\square$
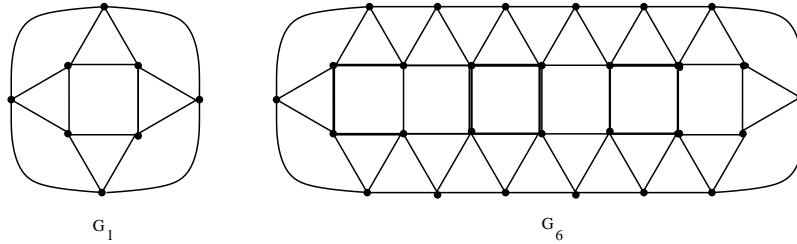

Figure 5.2; Family of graphs $\{G_i\}$, where i is the number of squares

We have constructed in this section, for any n, graphs which have $\mathbb{Z}_n$ in torsion of cohomology. In Corollary 4.10 we have proven that a simple graph

with a 3-cycle has $\mathbb{Z}_3$ torsion in cohomology. However Theorem 4.1 allows us to construct any torsion, even for a graph with no 3-cycle.

Corollary 5.2. *For any finitely generated abelian group T there is a simple graph, G, without any 3-cycle such that $tor H_{\mathcal{A}_3}^{1,2v-3}(G) = T$.*

Proof. For a simple graph, G without a 3-cycle, the torsion of $H_{\mathcal{A}_3}^{1,2v-3}(G)$ is equal to the torsion of $H_1(X_4(G)$ where, $X_4(G)$ is obtained from G by attaching a 2-cell along every 4-cycle. It is not difficult to construct G such that $tor X_4(G) = T$. For example, to obtain $T = \mathbb{Z}_2$ we divide a projective plane into sufficiently many squares (25 suffices). In Figure 5.3 we present a graph with no 3-cycles and $T = \mathbb{Z}_3$ (square divided into 6 by 6 small squares and boundary quotiented by $\mathbb{Z}_3$ action). $\square$
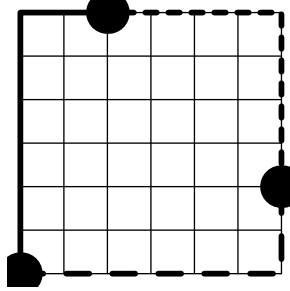

Figure 5.3: A graph with $tor H_{\mathcal{A}_3}^{1,63}(G) = \mathbb{Z}_3$

6. Examples of graphs with the same dichromatic (and Tutte) polynomial but different $\mathcal{A}_3$ first cohomology

In this section we construct graphs which have the same chromatic (even dichromatic and Tutte) polynomial but different $H_{\mathcal{A}_3}^{1,2v-3}(G)$. In relation to these examples we also describe A_3 graph cohomology for the one vertex product of graphs. We work also with the 2-vertex product of graphs, showing in particular that the Whitney flip (reglueing of vertices) preserves the cohomology of the product[10]. In effect, $H_{\mathcal{A}_3}^{1,2v-3}(G)$ is a 2-isomorphism (i.e. matroid) graph invariant.

One of the corollaries of our main theorem (Theorem 4.1) is that we can give simple formulas for cohomology of vertex products of graphs, $G * H$ and and edge products of graphs, $G|H$, (if working with $\mathbb{Z}_3$ coefficients), where $G|H$ is obtained from G and H by identifying an edge in G with an edge in H. An edge product depends on the choice of identified edges (see Fig. 6.1 for simple examples) but the chromatic polynomial $(P(G) \in \mathbb{Z}[\lambda])$ is always the same, $P(G|H) = P(G)P(H)/(\lambda(\lambda - 1))$ (see e.g. [Big]). However we can often differentiate between different products $G|H$ using $H_{\mathcal{A}_3}^{1,2v-3}(G|H)$, for example for the first pair in Figure 6.1 we get, $\mathbb{Z}_3^2 \oplus \mathbb{Z}$ and $\mathbb{Z}_3^2$, respectively and for the second pair in Figure 6.1 we get, $\mathbb{Z}_3^3 \oplus \mathbb{Z}^4$ and $\mathbb{Z}_3^3 \oplus \mathbb{Z}^3$,

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[10]Whitney flip of graphs is closely related to mutation of links.

respectively.
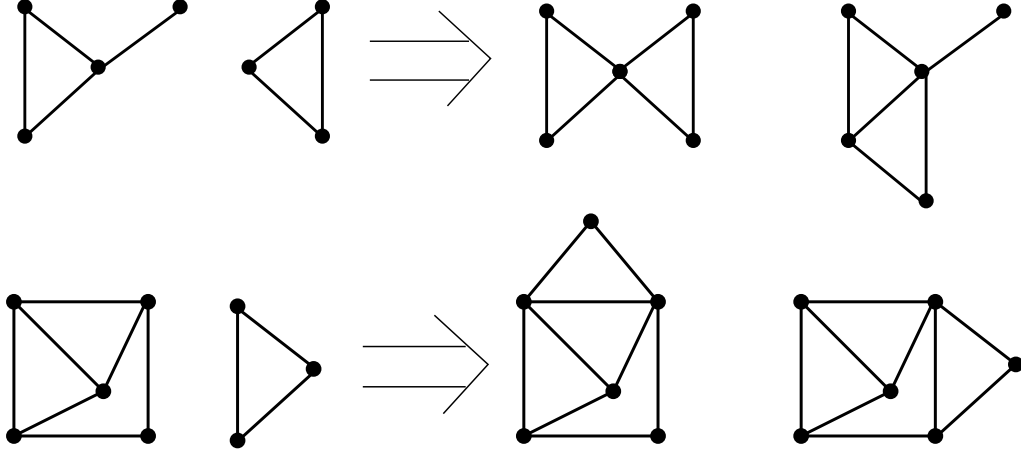
Edge products can give different graphs with the same chromatic polynomial

Figure 6.1:

In the 1930's Marion Cameron Gray found the following example of graphs which are not 2-isomorphic but have the same dichromatic (so also Tutte) polynomials, Figure 6.2, [Tut, Big]. These examples have different first $\mathcal{A}_3$ graph cohomology. In fact they differ from the second example in Figure 6.1 only by multiple edges, thus $H^{1,9}_{\mathcal{A}_3}(\mathrm{Gray}_1) = \mathbb{Z}_3^3 \oplus Z^4$ and $H^{1,9}_{\mathcal{A}_3}(\mathrm{Gray}_2) = \mathbb{Z}_3^3 \oplus Z^3$.
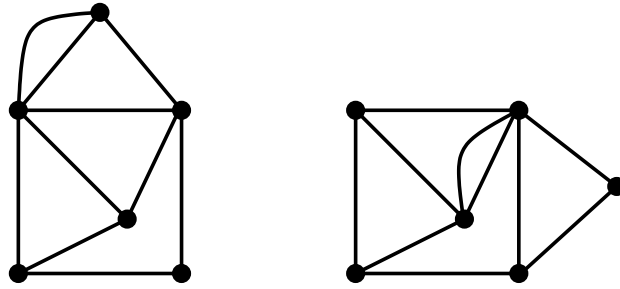
Codichromatic graphs of Marion C. Gray

Figure 6.2:

The Gray graphs are not 2-isomorphic (see Definition 6.1) however as an example of codichromatic graphs with different $\mathcal{A}_3$ graph cohomology they

are a little disappointing as they have multiple edges. In the sequel paper we will analyze examples obtained by rotation [Tut, APR].

We devote the rest of this section to the proof that 2-isomorphic graphs have isomorphic cohomology $H_{\mathcal{A}_3}^{1,2v-3}$. First recall definition of 2-isomorphism in the form convenient for our considerations, i.e. based on one-vertex, $G*H$, and two-vertex, G_*^*H, products.

Definition 6.1.   (1) *A one-vertex product $G * H = G * (v = w)H$ of graphs G and H with base points v and w respectively is obtained by gluing G with H by identifying v and w.*

(2) *We define a two-vertex product $G_*^*H = G_{*(v_2=w_2)}^{*(v_1=w_1)}H$ as follows. Given a graph G with two chosen vertices v_1, v_2 and a graph H with chosen vertices w_1 and w_2 we obtain a two-vertex product of G and H by identifying v_1 with w_1 and v_2 with w_2. If we switch the roles of w_1 and w_2 in a two-vertex product we obtain the new graph $G_{*(v_2=w_1)}^{*(v_1=w_2)}H$ (in standard terminology we say that these two graphs differ by Whitney flip).*

(3) *2-isomorphism of graphs is the smallest equivalence relation on isomorphism classes of graphs which satisfies:*
(i) One-vertex products of G and H (for any attachments) are 2-isomorphic.
(ii) Two graphs which differ by Whitney flip are 2-isomorphic.

Theorem 6.2.
(1) Let v and w be two vertices on the graph Y (not necessary connected), and let $Y' = Y/(v = w)$ be the graph obtained from Y by identifying v with w. If the distance $d_Y(v, w) > 4$ (allowing $d_Y(v, w) = \infty$), then $H_{\mathcal{A}_3}^{1,2v(Y')-3}(Y') = H_{\mathcal{A}_3}^{1,2v(Y)-3}(Y)$. In particular,

$$H_{\mathcal{A}_3}^{1,2v(G*H)-3}(G * H) = H_{\mathcal{A}_3}^{1,2v(G)-3}(G) \oplus H_{\mathcal{A}_3}^{1,2v(H)-3}(H).$$

(2) $H_{\mathcal{A}_3}^{1,2v(G_*^*H)-3}(G_{*(v_2=w_2)}^{*(v_1=w_1)}H) = H_{\mathcal{A}_3}^{1,2v(G_*^*H)-3}(G_{*(v_2=w_1)}^{*(v_1=w_2)}H)$.
(3) If two graphs, G_1 and G_2 are 2-isomorphic then $H_{\mathcal{A}_3}^{1,2v-3}(G_1) = H_{\mathcal{A}_3}^{1,2v-3}(G_2)$.

Proof. Without loss of generality we can assume that we deal only with simple graphs (in case when our operation produce a multiple edge we will replace it by a singular edge without affecting cohomology).
(1) It follows directly from Theorem 4.1: first the cell complex $X_{\Delta,3}(Y')$ is equal to $X_{\Delta,3}(Y)$. Furthermore, $t_2 - \frac{d_2}{2}$ and sq are the same for Y' and Y.
(2) We will consider three case depending on position of v_1, v_2, w_1 and w_2 in respective graphs. (i) We assume that either v_1 is connected by an edge to v_2 in G or w_1 is connected by an edge to w_2 in H.
(ii) Assume that (i) does not hold and the distance

$$d_G(v_1, v_2) + d_H(w_1, w_2) > 4.$$

(iii) Assume that neither (i) nor (ii) hold, that is,

$$d(v_1, v_2) = 2 = d(w_1, w_2).$$

First notice that

$$(t_2 - \frac{d_2}{2} - sq)(G_*^*H)$$
$$= (t_2 - \frac{d_2}{2} - sq)(G) + (t_2 - \frac{d_2}{2} - sq)(H) + \begin{cases} 0 & \text{in the cases (i), (ii)} \\ 1 & \text{in the case (iii)} \end{cases}$$

To complete our proof of the theorem we should show, according to Theorem 4.1, that $H^2(X_{\Delta,4}, \mathbb{Z})(G_*^*H)$ is invariant under Whitney flip. In the sequel paper we will discuss precise formulas relating $H^2(X_{\Delta,4}(G_*^*H), \mathbb{Z})$ with $H^2(X_{\Delta,4}, \mathbb{Z})(G)$ and $H^2(X_{\Delta,4}, \mathbb{Z})(G)$, compare Proposition 6.3 ; here we only notice that Whitney flip of G_*^*H changes the chain complex of cell-complex $X_{\Delta,4}(G_*^*H))$ only by isomorphism. We discuss concisely the cases (i), (ii) and (iii) separately.

(i) In this case we can think of $G_{*(v_2=w_2)}^{*(v_1=w_1)}H$ as being of the form $G|H$ were v_1, v_2 are connected by an edge, $\overrightarrow{e_G}$, in G and w_1, w_2 are connected by an edge, $\overrightarrow{e_H}$, in H. Then we check that constructing $G|H$ by glueing $\overrightarrow{e_G}$ to $\overrightarrow{e_H}$ or $\overrightarrow{e_G}$ to $\overleftarrow{e_H}$ yields the same cohomology. That is, chain complexes of $X_{\Delta,4}(G|(\overrightarrow{e_G} = \overrightarrow{e_H}H)$ and of $X_{\Delta,4}(G|(\overrightarrow{e_G} = \overleftarrow{e_H})$ are isomorphic (the isomorphism is sending $\overrightarrow{e}$ to $\overrightarrow{e}$ if $\overrightarrow{e}$ is an edge of G and it is sending $\overrightarrow{e}$ to $\overleftarrow{e}$ if $\overrightarrow{e}$ is an edge of H.

(ii) The distance $d(v_2, w_2)$ in $G * (v_1 = w_1)H$ is greater than 4. Therefore we can use (1) to conclude that

$$H_{\mathcal{A}_3}^{1,2v(G_*^*H)-3}(G_*^*H) = H_{\mathcal{A}_3}^{1,2v(G)-3}(G) \oplus H_{\mathcal{A}_3}^{1,2v(H)-3}(H).$$

(iii) In this case $d(v_1, v_2) = 2 = d(w_1, w_2)$, therefore v_1 is connected with v_2 in G by a "joint" $(e_G^{(1)}, e_G^{(2)})$ in G and by a "joint" $(e_H^{(1)}, e_H^{(2)})$ in H. As in the case (i) we can see that $X_{\Delta,4}(G_{*(v_2=w_2)}^{*(v_1=w_1)}H)$ and $X_{\Delta,4}(G_{*(v_2=w_1)}^{*(v_1=w_2)}H)$ yield isomorphic chain complexes of cell-complex homology. As in (i) we send $\overrightarrow{e}$ to $\overrightarrow{e}$ if $\overrightarrow{e}$ is an edge of G and we send $\overrightarrow{e}$ to $\overleftarrow{e}$ if $\overrightarrow{e}$ is an edge of H.

(3) It follows from (1) and (2) by definition of 2-isomorphism □

As mentioned before, we can use Theorems 4.1 and 4.2 to find the formulas for cohomology of an edge and 2-vertex products of graphs. We discuss these in a sequel paper, listing here three easy but useful special cases.

Proposition 6.3.

(i) *Let G and H be simple graphs and*
  $G|H = G|(\vec{e}_G = \vec{e}_H)H$, *where $\vec{e}_G$, $\vec{e}_H$ are identified edges. Then*

$$H_{\mathcal{A}_3}^{1,2v(G|H)-3}(G|H, \mathbb{Z}_3) = H_{\mathcal{A}_3}^{1,2v(G)-3}(G, \mathbb{Z}_3) \oplus H_{\mathcal{A}_3}^{1,2v(H)-3}(H, \mathbb{Z}_3) \oplus \mathbb{Z}_3^{t(e_G)t(e_H)},$$

  where $t(e_G)$ is the number of 3-cycles in G containing e_G.

(ii) $H_{\mathcal{A}_3}^{1,2v(G)-1}(G|P_3) = H_{\mathcal{A}_3}^{1,2v(G)-3}(G) \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^{t(e_G)}.$

(iii) $H_{\mathcal{A}_3}^{1,2v(G)}(G|P_4) = H_{\mathcal{A}_3}^{1,2v(G)-3}(G) \oplus \mathbb{Z}$.

Sketch of a proof of (i).

First we find $H_1(X_{\Delta,4}(G|H,\mathbb{Z}_3))$. As noted in Remark 4.4, $H_1(X_{\Delta,4}(G|H)) = H_1(X_{\Delta,4'}(G|H))$ and[11] in $X_{\Delta,4'}$ every 2-cell is glued either to G or to H. We know, Remark 4,4, that $H_1(X_{\Delta,4}(G|H)) = H_1(X_{\Delta,4'}(G|H))$ and $H_2(X_{\Delta,4}(G|H)) = H_2(X_{\Delta,4'}(G|H)) + \mathbb{Z}^{sq'(G|H)}$. Furthermore, if we work over $\mathbb{Z}_3$ we have $H_1(X_{\Delta,4'}(G|H),\mathbb{Z}_3) = H_1(X_{3,4'}(G|H),\mathbb{Z}_3)$ and $H_2(X_{\Delta,4'}(G|H),\mathbb{Z}_3) = H_2(X_{3,4'}(G|H),\mathbb{Z}_3) \oplus \mathbb{Z}^{t_3}$. Since $e_G = e_H$ is contractible in $\hat{X}_{3,4'}(G|H)$ we have $H_2(\hat{X}_{3,4'}(G|H),\mathbb{Z}_3) = H_2(\hat{X}_{3,4'}(G),\mathbb{Z}_3) \oplus H_2(\hat{X}_{3,4'}(H),\mathbb{Z}_3)$. These, taken together, give:

$H_2(X_{\Delta,4}(G|H),\mathbb{Z}_3) = H_2(X_{\Delta,4}(G),\mathbb{Z}_3) \oplus H_2(X_{\Delta,4}(H),\mathbb{Z}_3) + \mathbb{Z}_3^{t(e_G)t(e_H)}$

as $sq'(G|H) = sq'(G) + sq'(H) + t(e_G)t(e_H)$.

Example 6.4. *For graphs of Gray we have:* $H_{\mathcal{A}_3}^{1,9}(Gray) = \mathbb{Z}_3^3 \oplus Z^4$. *Each of these graphs is built as an edge product of the broken wheel* W_4^{in}, *Figure 8.3, triangle and doubling two of the edges. In homology we start from the group* $\mathbb{Z}_3^2 \oplus Z^4$. *Adding a triangle along an edge transforms it to* $\mathbb{Z}_3^3 \oplus Z^4$, *while doubling edges preserves it.*

7. Dichromatic graph cohomology

M. Stosic observed that chromatic graph cohomology can be slightly deformed to give a categorification of the dichromatic polynomial [Sto].

(i) We define the dichromatic cohomology $\tilde{H}_{\mathcal{A}_m}^{i,j}, \mathbb{M}(G)$ as follows:
We consider the algebra of truncated polynomials $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$

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[11]For the convenience of a reader we recall useful definitions of various cell-complexes built from G:

1(i) $\hat{X}_3(G)$ is the cell complex obtained from G by attaching 2-cells along 3-cycles in G,

1(ii) $\hat{X}_{3,4'}(G)$ is the cell complex obtained from $\hat{X}_3(G)$ by attaching 2-cells along 4-cycles with no diagonals.

1(iii) $\hat{X}_{3,4'}(G)$ is the cell complex obtained from $\hat{X}_3(G)$ by attaching 2-cells along 4-cycles.

2(i) $\hat{X}_{(3)}(G)$ is the cell complex obtained from G by identifying edges of every triangle in a coherent way (i.e. $\vec{e}_i = \vec{e}_j = \vec{e}_k$ for a triangle oriented as in Figure 4.8).

2(ii) $\hat{X}_{(3),4'}(G)$ is the cell complex obtained from $\hat{X}_{(3)}(G)$ by attaching 2-cells along 4-cycles with no diagonals.

2(iii) $\hat{X}_{(3),4}(G)$ is the cell complex obtained from $\hat{X}_{(3)}(G)$ by attaching 2-cells along 4-cycles.

3. If $\hat{X}(G)$ denotes one of cell complexes defined in 1 or 2 the $X(G)$ is obtained from $\hat{X}(G)$ by identifying all vertices of G.

4(i) $X_\Delta(G)$ is a cell complex obtained from G by identifying vertices of G and then adding 2-cells along expressions $2\vec{e}_3 - \vec{e}_2 - \vec{e}_1$ for any 3-cycle in G – two 2-cells added per every 3-cycle (see Fig. 4.7); compare Footnote 7.

4(ii) $X_{\Delta,4'}(G)$ is the cell complex obtained from $X_\Delta(G)$ by attaching 2-cells along 4-cycles with no diagonals.

4(iii) $X_{\Delta,4}(G)$ is the cell complex obtained from $X_\Delta(G)$ by attaching 2-cells along 4-cycles.

and $M = \mathcal{A}_m$ or $M = (x^{m-1})$. We modify d_e (from the definition of the chromatic chain complex) to get the chain complex $C^{*,*}_{\mathcal{A}_m, \mathbb{M}}(G)$. The only modification is in the case e has endpoints on the same component of $[G : s]$. We put $\tilde{d}_e = x^{m-1} Id$. The grading which Stosic associates to x^i is equal to $m - 1 - i$ and the grading of an element $(x^{i_0}, x^{i_1}, ..., x^{k(s)-1})$ in $C^i_{\mathcal{A}_m}$ is given by $i + k(s)(m-1) - \sum i_j$. Notice that for $i < \ell(G)$ we have $(i + k(s) = v$. In this way we have graded cochain complex and graded cohomology which can be used to recover dichromatic polynomial (so it is named *dichromatic graph cohomology*).

(ii) It follows from the definition that for $i < \ell(G)$ we have $d^i = \tilde{d}^i$. Therefore, for $i < \ell(G) - 1$, $\tilde{H}^i_{\mathcal{A}_m}(G) = H^i_{\mathcal{A}_m}(G)$. More precisely, $\tilde{H}^{i, v(m-1)-j}_{\mathcal{A}_m}(G) = H^{i,j}_{\mathcal{A}_m}(G)$. Furthermore for $i = \ell(G) - 1$, we have: $tor(\tilde{H}^{i, v(m-1)-j}_{\mathcal{A}_m}(G)) = tor(H^{i,j}_{\mathcal{A}_m}(G))$.

Most of our computations done of $H^{*, 2v-3}_{\mathcal{A}_3}(G)$ is also valid for dichromatic graph cohomology (one should remember about grading transformation).

8. Computational results, width of cohomology

In previous sections, we completely computed $H^{1, v-1}_{\mathcal{A}_2}(G)$ and $H^{1, 2v-3}_{\mathcal{A}_3}(G)$. Here we present a few computational results and conjectures derived from them for $\mathcal{A}_m$ graph cohomology, for $m \geq 3$. In the first subsection we still work over $\mathcal{A}_3$ algebra searching for gradings with non-trivial torsion.

8.1. Width of $tor(H^{1,*}_{\mathcal{A}_3}(G))$.

Let j_{max} be the maximal index j such that $tor H^{1,j}_{\mathcal{A}_m}(G) \neq 0$ and j_{min} be the minimal index j such that $tor H^{1,j}_{\mathcal{A}_m}(G) \neq 0$. We define the width, $w^1_{\mathcal{A}_m}(G)$ of the torsion $tor H^{1,*}_{\mathcal{A}_m}(G)$ as follows:

$$w^1_{\mathcal{A}_m}(G) = \begin{cases} -1 & \text{if } H^{1,*}_{\mathcal{A}_m}(G) \text{ are all free} \\ j_{max} - j_{min} & \text{otherwise} \end{cases}$$

As we mentioned before for an odd n, $tor H^{1,*}_{\mathcal{A}_m}(P_n) = H^{1, \frac{n-1}{2}m}_{\mathcal{A}_m}(P_n) = \mathbb{Z}_m$ thus the width $w^1_{\mathcal{A}_m}(P_n) = 0$. We also know that for any graph and algebra $\mathcal{A}_3$, $j_{max} \leq 2v - 3$ (Proposition 2.8). In fact, from results of Section 4 it follows that for a simple graph G, $j_{max} = 2v - 3$ iff G contains a triangle or $H_1(X_4(G))$ has a torsion. In this section we analyze $tor H^{1,*}_{\mathcal{A}_3}(G)$ (in particular j_{min}) for several families of graphs and conjecture a formula for width.

Let $P_{t,k} = P_3 | P_k$ be a graph with $k + 1$ vertices obtained by gluing a triangle and a k-gon along an edge, Figure 8.1.
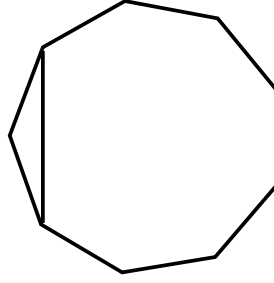
Figure 8.1: $P_{t,8} = P_3 | P_8$

We know that $H_{\mathcal{A}_3}^{1,2k-1}(P_{t,k}) = \mathbb{Z}_3$ for $k > 4$ ($H_{\mathcal{A}_3}^{1,5}(P_{t,3}) = \mathbb{Z}_3^2 \oplus \mathbb{Z}$ and $H_{\mathcal{A}_3}^{1,7}(P_{t,4}) = \mathbb{Z}_3 \oplus \mathbb{Z}$). Computational results are summarized in the following Table and conjecture:

Conjecture 8.1. *For graphs $P_{t,k} = P_3 | P_k$ the following is true:* $w_{\mathcal{A}_3}^1(P_{t,k}) = k - 3$

k	$P_{t,k}$	$tor H_{\mathcal{A}_3}^{1,2v-7}$	$tor H_{\mathcal{A}_3}^{1,2v-6}$	$tor H_{\mathcal{A}_3}^{1,2v-5}$	$tor H_{\mathcal{A}_3}^{1,2v-4}$	$H_{\mathcal{A}_3}^{1,2v-3}$
3		0	0	0	0	$\mathbb{Z}_3^2 \oplus \mathbb{Z}$
4		0	0	0	$\mathbb{Z}_3^2$	$\mathbb{Z}_3 \oplus \mathbb{Z}$
5		0	0	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^4$	$\mathbb{Z}_3$
6		0	$\mathbb{Z}_3$	$\mathbb{Z}_3^5$	$\mathbb{Z}_3^4$	$\mathbb{Z}_3$
7		$\mathbb{Z}_3$	$\mathbb{Z}_3^6$	$\mathbb{Z}_3^{10}$	$\mathbb{Z}_3^5$	$\mathbb{Z}_3$
8		$\mathbb{Z}_3^6$	$\mathbb{Z}_3^{15}$	$\mathbb{Z}_3^{14}$	$\mathbb{Z}_3^6$	$\mathbb{Z}_3$

Another family of graphs we consider, G_{t,s^k}, is obtained by glueing a triangle and k squares ($v = 3 + 2k$ vertices)in a sequence along one edge; Figure 8.2.
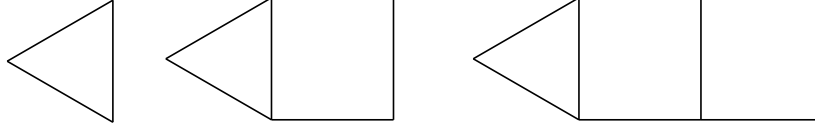

Figure 8.2: Family of G_{t,s^k} graphs: $G_{t,s^0} = P_3$, $G_{t,s}$ and G_{t,s^2}

From the theory developed earlier we get: $H_{\mathcal{A}_3}^{1,4k+3}(G_{t,s^k}) = \mathbb{Z}_3 \oplus \mathbb{Z}^k$ for $k \geq 0$. Computational results are presented in the following conjectures and Table.

Conjecture 8.2. $torH_{\mathcal{A}_3}^{1,4k+2}(G_{t,s^k}) = \mathbb{Z}_3^{2k}$.

Conjecture 8.3. $torH_{\mathcal{A}_3}^{1,3k+3}(G_{t,s^k}) = \mathbb{Z}_3^{2^k}$.

Conjecture 8.4. $w_{\mathcal{A}_3}^1(G_{t,s^k}) = k$

k	G_{t,s^k}	$torH_{\mathcal{A}_3}^{1,2v-6}$	$torH_{\mathcal{A}_3}^{1,2v-5}$	$torH_{\mathcal{A}_3}^{1,2v-4}$	$H_{\mathcal{A}_3}^{1,2v-3}$
1		0	0	$\mathbb{Z}_3^2$	$\mathbb{Z}_3 \oplus \mathbb{Z}$
2		0	$\mathbb{Z}_3^4$	$\mathbb{Z}_3^4$	$\mathbb{Z}_3 \oplus \mathbb{Z}^2$
3		$\mathbb{Z}_3^8$	$\mathbb{Z}_3^{12}$	$\mathbb{Z}_3^6$	$\mathbb{Z}_3 \oplus \mathbb{Z}^3$
4		$\mathbb{Z}_3^{32}$	$\mathbb{Z}_3^{24}$	$\mathbb{Z}_3^8$	$\mathbb{Z}_3 \oplus \mathbb{Z}^4$

Next, we consider family of wheels W_k, that is, cones over $(k-1)$-gones. $H^{1,2k-3}$ was computed in Corollary 4.11 and computational results presented in the next Table imply the following conjecture:

Conjecture 8.5. $w_{\mathcal{A}_3}^1(W_n) = [\frac{n-3}{2}]$

n	$tor H_{\mathcal{A}_3}^{1,2v-6}(W_n)$	$tor H_{\mathcal{A}_3}^{1,2v-5}(W_n)$	$H_{\mathcal{A}_3}^{1,2v-4}$	$H_{\mathcal{A}_3}^{1,2v-3}(W_n)$
4	0	0	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}^2$
5	0	0	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3 \oplus \mathbb{Z}^5$
6	0	0	$\mathbb{Z}_3^5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^5 \oplus \mathbb{Z}^5$
7	0	$\mathbb{Z}_3^2$	$\mathbb{Z}_3^9$	$\mathbb{Z}_3^5 \oplus \mathbb{Z}^7$
8	0	$\mathbb{Z}_3^7$	$\mathbb{Z}_3^{14}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^7 \oplus \mathbb{Z}^7$
9	$\mathbb{Z}_3^5$	$\mathbb{Z}_3^{20}$	$\mathbb{Z}_3^{21}$	$\mathbb{Z}_3^7 \oplus \mathbb{Z}^9$

Examples, given above, may suggest that if a simple graph G has a cycle of odd length n then the group $H_{\mathcal{A}_3}^{1,2(v-1)-\frac{n-1}{2}}(G)$ has torsion. However we computed that the only torsion in $H_{\mathcal{A}_3}^{1,j}(K_4)$ is supported by $j = 5$ and the only torsion in $H_{\mathcal{A}_3}^{1,j}(K_5)$ is supported by $j = 7$, despite the fact that K_4 and K_5 have 5-cycles. In fact, calculations of $tor H_{\mathcal{A}_3}^{1,j}(K_n)$ suggest:

Conjecture 8.6. $w_{\mathcal{A}_3}^1(K_n) = 0$

8.2. **Computation over the $\mathcal{A}_5$ algebra: $H_{\mathcal{A}_5}^{1,4v-7}(G)$.** So far, according to calculations for different series of graphs (wheels, broken wheels and complete graphs) the only torsion is $\mathbb{Z}_2$ and $\mathbb{Z}_5$. Furthermore, the rank of $\mathbb{Z}_5$ torsion is always the number of 3-cycles in a graph, while, for complete graphs, the rank of $\mathbb{Z}_2$ is the number of 4-cycles in a graph. We should stress that these results are not sufficient for general conjectures- even for $m = 3$ the above series of graphs have only $\mathbb{Z}_3$ and $\mathbb{Z}_2$ torsion but as we proved in Section 5 in $\mathcal{A}_3$- cohomology any torsion is possible.
First we consider a family of wheels, W_n and broken wheels: W_n^{out} is obtained from W_n by deleting an edge from the polygonal part of the wheel (treated as a cone over $(n-1$-gon), W_n^{in} is obtained from W_n by deleting a "spike" edge, Figure 8.3.

Conjecture 8.7. *For $n > 4$ the following holds[12]:*

$$H_{\mathcal{A}_5}^{1,4n-3}(W_n^{out}) = \mathbb{Z}_5^{n-1} \oplus \mathbb{Z}^{n-2}$$

$$H_{\mathcal{A}_5}^{1,4n-3}(W_n) = \mathbb{Z}_5^n \oplus \mathbb{Z}^n$$

$$H_{\mathcal{A}_5}^{1,4n-3}(W_n^{in}) = \mathbb{Z}_5^{n-2} \oplus \mathbb{Z}^{n-2}$$

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[12] $H_{\mathcal{A}_5}^{1,9}(W_3^{in}) = \mathbb{Z}_5^2 \oplus \mathbb{Z}$, $H_{\mathcal{A}_5}^{1,13}(W_4^{in}) = \mathbb{Z}_5^2 \oplus \mathbb{Z}^2$
$H_{\mathcal{A}_5}^{1,9}(W_3) = \mathbb{Z}_5^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^2$, $H_{\mathcal{A}_5}^{1,13}(W_4) = \mathbb{Z}_5^4 \oplus \mathbb{Z}^5$.
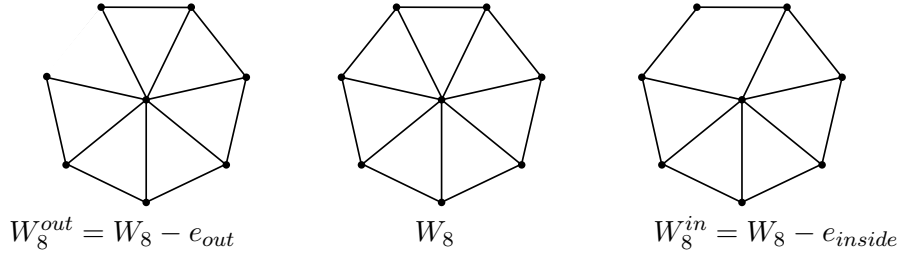
$W_8^{out} = W_8 - e_{out}$ W_8 $W_8^{in} = W_8 - e_{inside}$

Figure 8.3: Wheels and broken wheels

Conjecture 8.8. *For algebra $\mathcal{A}_5$ and complete graphs K_n with n vertices the following is true:*

$$H_{\mathcal{A}_5}^{1,4n-7}(K_n) = \mathbb{Z}_5^{\binom{n}{3}} \oplus \mathbb{Z}_2^{\binom{n}{4}} \oplus \mathbb{Z}^{2\binom{n}{4}}.$$

8.3. **Calculation of $H_{\mathcal{A}_m}^{1,(m-1)(v-2)+1}(G)$.**
Based on computational results we conjecture that the cohomology $H_{\mathcal{A}_m}^{1,,3m-2}(G)$
for several graphs including a square with a diagonal $(P_3|P_3)$, a "house"
$(P_3|P_4)$, K_4 and K_5.

Conjecture 8.9. *For $m > 2$ we have[13]: $H_{\mathcal{A}_m}^{1,2m-1}(P_3|P_3) = \mathbb{Z}_m^2 \oplus \mathbb{Z}$.*

We checked the conjecture for $m \le 15$

Conjecture 8.10. *For $m \ge 2$ we have $H_{\mathcal{A}_m}^{1,3m-2}(P_3|P_4) = \mathbb{Z}_m \oplus \mathbb{Z}$.*

We checked the conjecture for $m \le 14$.

Conjecture 8.11. *$(\forall m \ge 4)$ $H_{\mathcal{A}_m}^{1,2m-1}(K_4) = \mathbb{Z}_m^4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^2$.*

We checked the conjecture for $m \le 14$.

Conjecture 8.12. $H_{\mathcal{A}_m}^{1,3m-2}(K_5) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}^5, & m=2; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}^{10}, & m=3; \\ \mathbb{Z}_2^5 \oplus \mathbb{Z}_m^{10} \oplus \mathbb{Z}^{10}, & m=4; \\ \mathbb{Z}_2^5 \oplus \mathbb{Z}_m^{10} \oplus \mathbb{Z}^{10}, & m > 4, m \text{ odd}; \\ \mathbb{Z}_2^{10} \oplus \mathbb{Z}_m^{10} \oplus \mathbb{Z}^{10}, & m > 5, m \text{ even}. \end{cases}$

We checked the conjecture for $m \le 9$.
   We hope that conjectures and calculations listed in this Section will en-
courage future research.

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[13]$H_{\mathcal{A}_2}^{1,3}(P_3|P_3) = \mathbb{Z}_2 \oplus \mathbb{Z}$.

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